Heterogeneous Beliefs and Business Cycles *

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Abstract

We study heterogeneous beliefs about TFP growth in a complete-market production economy where employment is hired in advance. The firm's discount factor inherits a wealth-weighted average of investor beliefs. Waves of optimism ripple into the firm's investment in hours thus tying together the equity premium and labor volatility puzzles. We present a taxonomy of beliefs that shows the implications of different belief models for asset prices and business cycles. We argue that when beliefs are extrapolative, they add volatility to asset prices and labor markets, contributing to the resolution of both puzzles. With extrapolative beliefs, the length of a cycle is correlated with its amplitude and stock-market turnover is countercyclical. The model is consistent with several asset-pricing and business cycle moments.

KEYWORDS: Heterogeneous Beliefs; Business Cycles; Asset Prices; Speculation.

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1 Introduction

One of the most prominent economic narratives is that severe downturns are driven by swings in investor sentiments. Tracing back to the works of Hyman Minsky, Charles Kindleberger, and Robert Shiller—e.g., Kindleberger and Aliber (2015) and Shiller (2000)—we find a common description: fueled by leverage, a wave of excessive optimism about stock earnings prompts high stock prices. High stock prices, in turn, stimulate firm investments and hiring. Periods of over-optimism end with disappointing earnings news that bursts asset prices on Wall Street. The blast wave is felt on Main Street.

Theoretically, Harrison and Kreps (1978) and Scheinkman and Xiong (2003) formalize the notion that asset prices can be driven by optimism waves. These papers also show that belief heterogeneity is necessary to induce leverage which can further fuel asset prices. A virtue of these models is that they subtly depart from the discipline imposed by rational expectations: all agents understand the workings of the economy, but some are simply more optimistic or pessimistic than others. While these models lay out the necessary ingredients to study how beliefs affect asset prices and leverage, it is still unclear how and by how much can waves of optimism percolate into the real economy.

This paper studies a benchmark economy where beliefs are a direct source of business cycle fluctuations. This benchmark economy has no financial frictions, prices are flexible, and markets are complete. The only link between beliefs and output is labor hiring risk. Because hiring is decided before firms know their productivity, hours fluctuations are driven exclusively by investor beliefs about earnings growth and risk appetite.

A growing literature is studying how beliefs impact the real economy by aggravating nominal rigidities or financial constraints. However, to the best of our knowledge, we do not have a frictionless benchmark economy where beliefs are a direct source of business cycle fluctuations. Developing a frictionless benchmark economy with belief heterogeneity is important. First, by abstracting away market imperfections, we can provide a qualitative and quantitative assessment of the direct effects of beliefs on the business cycle. Second, we can derive general principles that extend to environments where market imperfections amplify direct effects. Finally, a frictionless benchmark describes an ideal economy that policies aimed at offsetting market imperfections should attempt to mimic.

The Model. In the model, belief heterogeneity impacts hiring decisions. A representative firm hires labor. Households agree to disagree about the evolution of total factor productivity (TFP) growth. Depending on their views, households buy and sell shares of a representative firm. We distinguish between risk aversion and elasticities of intertemporal substitution by endowing households with Epstein-Zin preferences. Households supply labor that enters a GHH static utility bundle, as in Greenwood et al. (1988).

Beliefs govern the firm's labor demand because hiring is risky. As in Burnside et al. (1993), labor is chosen prior to the realization of shocks. This timing induces an operational leverage channel whereby hiring becomes a risky investment, as occurs also in labor search models.[1] This feature is important to produce business-cycle fluctuations from asset-price fluctuations. We focus on the firm's investment in forming a workforce, as opposed to physical capital formation, because it is understood that fluctuations in capital investment cannot be a source of business cycles (Chari et al., 2007).[2]

Belief heterogeneity is essential to inducing leverage and turnover dynamics, recurrent themes in the optimism narratives. As a result of belief heterogeneity, households hold long or short positions in firm shares. More optimistic households lever up their stock holdings.

Leverage produces an internal propagation mechanism as it induces differences in risk exposures. More optimistic agents generate capital gains when good states are realized and vice-versa. This feature produces internal propagation because wealth-weighted beliefs affect asset prices. For example, when wealth-weighted beliefs are more optimistic a greater demand for risky assets puts upward pressure on stock prices and compresses risk premia.

In the environment, movements in discount rates affect firms' decisions. Given that hiring is risky, as investors' risk appetite increases, discount rates fall. This stimulates hiring.[3] All in all, the equity premium, the equity volatility puzzle, and *labor volatility* puzzles are tied together into a single puzzle.

Theoretical Results. Despite featuring Epstein-Zin preferences and an endogenous labor supply, the model is highly tractable thanks to an "as if" property. Namely, the model can be solved as if it were an endowment economy that shares the same household stochastic discount factor (SDF) as the original economy. This property renders a tractable characterization. Importantly, the firm making risky hiring decisions in the original econ-

[1] We reinterpret this form of *investment in labor* as a simplified version of the investment in hiring that happens in labor-search models. Recall that in labor-search models, firms incur in sunk costs to hire workers and, thus, there is a force toward long-lasting relationships.

[2] Investment is small relative to the capital stock in a business cycle model. Thus, fluctuations in investment do not meaningfully impact the production possibility frontier.

[3] This mechanism is grounded on evidence that risk-adjusted excess returns are high in recessions (Lustig and Verdelhan, 2012). Lustig and Verdelhan (2012) emphasizes that due to higher capital costs, even unconstrained firms cut back investment and hiring during recessions. Hall (2017) puts a similar mechanism to work in a Diamond-Mortensen-Pissarides framework.

omy uses the common SDF to put weight across states. Thus, the economy's SDF maps into a labor-demand factor that, ultimately, determines labor and output. This mapping transparently demonstrates how the evolution of wealth-weighted beliefs jointly affects asset prices and output. Furthermore, we do not make any assumptions regarding the firm's information about investor beliefs; the firm only needs to understand how its hiring decisions affect its stock value. The solution approach may be convenient in other environments.

When we specialize Epstein-Zin preferences to log utility, the model features an analytic solution. This analytic solution allows us to articulate a belief taxonomy. Namely, we catalog beliefs into two broad categories with two sub-categories: One category is rank-preserving beliefs—which we sub-catalog into optimistic and pessimistic beliefs. The other category are rank-alternating beliefs—which we sub-catalog into extrapolative and intrapolative beliefs. Under this taxonomy, we provide general business-cycle properties for each belief system.

We uncover several principles regarding the amplification properties of different belief systems. The first principle is that only extrapolative beliefs amplify business cycles in all states—relative to rational expectations. Amplification is provoked by beliefs that boost the firm's discount factor during booms but depress this discount factor during busts. Only extrapolative have this property. Only for this belief system do agents think that current states are more persistent than they actually are.

A second principle is that only extrapolative beliefs engender *risk build-up*. That is, when a fraction of households is extrapolative, the longer the boom, the deeper the bust. This property is exclusive to extrapolative beliefs because extrapolative households consistently accumulate wealth during the length of the boom. This wealth accumulation by extrapolative agents fuels the demand for risky assets and stimulates hiring. When the economy switches state, extrapolative agents remain relatively wealthier while at the same time becoming the most pessimistic. This change switch in optimism by the wealthiest agents amplifies the rift in the economy.

A third property regards stock-market turnover, a feature that receives little attention in the macro-finance literature. We show that when some households are extrapolative, there is a positive correlation between stock-market turnover and measured disagreement. Only for this class of beliefs is turnover particularly large when the economy enters a recession, a property that we verify holds in the data.

The taxonomy of beliefs showcases the importance of studying belief heterogeneity: evidence by López-Salido et al. (2017) and Krishnamurthy and Muir (2017) shows that risk premia are low during credit booms that are followed by subsequent crashes.

However, without belief heterogeneity macro-finance models generate the opposite prediction—i.e., Brunnermeier and Sannikov (2014a) and He and Krishnamurthy (2013). Furthermore, these properties are useful to distinguish models with heterogeneity in beliefs from models with heterogeneity in risk aversion. The inability to distinguish belief from risk heterogeneity is a common critique of belief models. We contend that heterogeneity in risk aversion cannot produce risk build up nor increased turnover during busts.

Quantitative Analysis. We conclude the paper with a quantitative evaluation of the theory. To properly measure belief heterogeneity in the data, we exploit survey data on earnings forecasts and estimate an econometric model to infer a distribution of beliefs. We find evidence of substantial heterogeneity in beliefs and use the estimated distribution of beliefs in our quantitative evaluation.

We calibrate the rest of the model following standard parameterizations. We then use the model to perform counterfactuals across different belief systems and levels of heterogeneity. We show that extrapolative beliefs are key to producing a large equity premium and volatile returns, while jointly generating substantial fluctuations in hours. The version with rational expectations fails on both dimensions. We show that by turning heterogeneity off, the model fails in other moments.

We finally provide an external validation of the mechanism. First, we construct a time series for a disagreement index obtained from our distribution of beliefs. We also show that consistent with our model, this index is correlated with stock-market turnover. We then confirm that average investor beliefs are correlated with firm-level hiring decisions, using a similar approach to Gennaioli et al. (2016). Finally, we show that our model delivers substantial predictability of asset prices providing a reinterpretation of the Campbell-Shiller decomposition.

We conclude by discussing some avenues for extensions. In the next section, we discuss how this paper fits in the literature and then move to the body of the paper.

Connection to the Literature. This paper is related to business cycle and asset-pricing analysis that originates with Kydland and Prescott (1982) and Mehra and Prescott (1985), respectively. As in our paper, a recent strand links fluctuations in risk premiums to real business cycles. Hall (2017), Borovička and Borovičková (2019) and Kehoe et al. (2019) explore a similar transmission mechanism to study unemployment fluctuations in the context of the Diamond-Mortensen-Pissarides search model. Di Tella and Hall (2019) stress the role of uninsurable idiosyncratic risk and precautionary savings.

Our paper also fits into the recent macro-finance literature that emphasizes the importance of the wealth share of special individuals (e.g., financial intermediaries) for the business cycle. For example, He and Krishnamurthy (2011), Brunnermeier and Sannikov (2014b), Mendo (2018), and Silva (2019), among others.

In our case, the wealth shares of non-rational investors is key. Relatedly, Detemple and Murthy (1994), Xiong and Yan (2009), Kubler and Schmedders (2012), among others, study how redistribution among agents due to heterogeneous beliefs affects asset price fluctuations. More recently, Caballero and Simsek (2020a) and Caballero and Simsek (2020b) show how financial trading between optimistic and pessimistic investors, by affecting the evolution of the distribution of wealth among them, amplifies a recession generated by a decline in risky asset valuations when output is determined by aggregate demand. In contrast, we focus on the supply-side of the economy, and explore the role of time-to-build as the key ingredient that connects beliefs and output fluctuations. We also study a wide taxonomy of beliefs.

In our quantitative exploration, we assume that some agents hold diagnostic beliefs in the spirit of Gennaioli and Shleifer (2010), this paper is also related to the literature that explores how subjective beliefs affect the business cycle. See, for example, Eusepi and Preston (2011), Angeletos et al. (2018), Bordalo et al. (2018), and Bhandari et al. (2019), among others. Relatedly, Adam and Merkel (2019) show that (homogeneous) extrapolative beliefs can explain the stock price and business cycles altogether. Both cycles are connected as high stock prices signal profitable investment opportunities to capital producers. In contrast to our work, these papers abstract from the role of heterogeneous beliefs.

Using multiple surveys of investor expectations, Greenwood and Shleifer (2014) provide evidence from multiple surveys that investors tend to extrapolate their expectations from the realization of stock returns. These surveys seem to capture fundamental behavior: investor surveys correlate stock market turnover (Greenwood and Shleifer, 2014), the cyclicality of credit and leverage (López-Salido et al., 2017), and correlate with firm-level investment decision Gennaioli et al. (2016).

Importantly, De La O and Myers (2021) provided evidence that earnings growth expectations are much more volatile than the volatility of returns. The latter finding contrasts with asset-pricing models because it suggests that the volatility of earnings expectations accounts for a large fraction of the volatility of the price-dividend rate. We interpret the evidence of De La O and Myers (2021) as an upper bound on the importance of beliefs. To see this, if investors have irrational beliefs—they make forecast errors—and are all identical, homogeneous $\mathbb{E}_t^m = \mathbb{E}_t^i$, and hence, beliefs can drive asset prices. However, if beliefs are heterogeneous, the contribution of forecast errors will critically depends on how representative are surveys of market-based expectations, $\mathbb{E}_t^m = \mathbb{E}_t^i$. In particular, survey data is not weighted by wealth.

There is also a tradition on heterogeneous beliefs, speculative behavior and bubbles, once short-selling constraints and alternating beliefs are accounted for, as in Harrison and Kreps (1978) and Scheinkman and Xiong (2003).[4] The interaction with financial markets is explored by Geanakoplos (2003, 2010), Fostel and Geanakoplos (2008), Simsek (2013a), Iachan et al. (2019), among others.

Some papers have studied different transmission mechanisms of speculative behavior and bubbles to the real sector. In Gilchrist et al. (2005), monopolistic firms can overcome short-selling by issuing shares at a price above fundamental value, which lowers the cost of capital and enhances investment. Bolton et al. (2006) present an agency model in which over-investment occurs during a bubble episode due to stock-based executive optimal compensation contracts that emphasize short-term stock performance. In contrast, Panageas (2005) shows that once investment subject to quadratic costs is introduced in a model with heterogeneous beliefs and a short-selling constraint, despite the speculative behavior of agents, the neoclassical q theory of investment remains valid. Related to our work, Buss et al. (2016) study policy implications in a quantitative framework in which agents trade for risk-sharing and speculative reasons, and speculation reduces investment and welfare by pushing the cost of capital up. As opposed to our work, these papers focus on investment in capital rather than hours, and not all of them feature models that are amenable to quantitative exercises.

Finally, our paper is also related to the natural selection literature, which asks whether those agents with incorrect beliefs eventually disappear. Blume and Easley (1992, 2006) and Sandroni (2000) argue that only those with more accurate beliefs survive in the long run in an environment with complete markets and separable preferences. However, this result is not robust to the market structure, as shown by Beker and Chattopadhyay (2010), Blume et al. (2018) and Cao (2018), and also not robust to preferences that are non-separable recursive even when markets are complete, as shown recently by Dindo (2019) and Borovička (2020). Closely related is Cao (2018), who works out the same investor problem as ours but does not link beliefs to TFP shocks in an RBC economy. In fact, the paper studies the natural selection hypothesis in an endowment economy with

[4]A large literature studies other types of bubbles that emerge for reasons other than heterogeneous beliefs, such as the so-called "rational bubbles" (Blanchard and Watson, 1982; Santos and Woodford, 1997). Martin and Ventura (2012) and Miao and Wang (2018) provide environments in which the collapse of rational bubbles leads to a recession. Other recent contributions emphasize the interaction of rational bubbles and policy, for example, Galí (2014), Hirano et al. (2015), Allen et al. (2018), and Asriyan et al. (2019). We leave the study of the role of policy in versions of our framework that also allow for bubbles for future research.

incomplete markets.[5]

Organization. Section 2 lays out the environment and introduces the main ingredients. Section 3 proceeds with the characterization of the equilibrium. Section 4 discusses a special case where the solution can be obtained in closed form. Section 5 presents the quantitative results. Section 6 concludes. All proofs of formal results are contained in the appendix.

2 Model

2.1 Environment

We consider a two-state complete-markets economy with time indexed by $t \in \{0, 1, ..., \}$. The economy is populated by heterogeneous households that differ in their beliefs regarding the growth of future TFP. Households hold (or issue) risk-free bonds and hold (or short-sell) shares of a single representative firm. Differences in beliefs induce a desire to lever. The firm hires labor one period in advance, prior to the realization of TFP. This timing for hires links asset prices with labor demand.

The exogenous state. Total factor productivity A_t grows according to a two-state Markov process:

$$\frac{A_{t+1}}{A_t} = x_{t+1}, \tag{1}$$

where $x_{t+1} \in \{x_L, x_H\}$, $0 < x_L < x_H$. The transition probabilities from state s to s' are denoted by $\{p_{ss'}\}$.

The firm. The representative firm produces a final good according to $A_{t+1}h_{t+1}^{\alpha}$, where labor h_{t+1} is hired in period t, prior to the realization of x_{t+1}. While firms hire and contract the wage W_{t+1} one period ahead, the wage bill is paid when production is finished.

The firm takes hours at the initial date h_0 as given and hires labor in subsequent periods to maximize its value using a stochastic discount factor (SDF), $\Lambda_{t,t+1}$:

$$Q_t = \max_{h_{t+1}} \mathbb{E}_t \left[\Lambda_{t,t+1} \left(\pi_{t+1} + Q_{t+1}\right)\right]. \tag{2}$$

[5] Below we confirm the natural selection hypothesis in an example with separable preferences and without short-selling constraints. In addition, despite these recent contributions, in all simulations reported in the paper, rational investors eventually accumulate the entire stock of investors' wealth.

Q_t denotes the firm value and $\pi_{t+1} \equiv A_{t+1}h^{\alpha}_{t+1} - W_{t+1}h_{t+1}$ denotes the profit (dividend). Expectations are taken with respect to the transition probabilities $\{p_{ss'}\}$ and weighted by $\Lambda_{t,t+1}$. Because markets are complete, there is a unique SDF. Hence, there is unanimity regarding the firm's objective among shareholders, as we discuss in Section 3.1. Beliefs affect employment decisions through their impact on the SDF.

Households. There is a finite number of infinite-lived investor households, indexed by $i \in \mathcal{I} = \{1, \ldots, I\}$ with masses $\{\mu_i\}$, $\sum_i \mu_i = 1$. Household i derives utility from consumption $C_{i,t}$ and disutility from working $h_{i,t}$. They have Epstein-Zin preferences over a GHH consumption-labor composite:

$$V_{i,t} = (1-\beta)U\left(C_{i,t} - \xi_t \frac{h^{1+\nu}_{i,t}}{1+\nu}\right) + \beta U\left(\mathcal{V}_{i,t}\right), \tag{3}$$

where $V_{i,t}$ denotes the utility level, β the discount factor, and ξ_t controls the labor disutility. $\mathcal{V}_{i,t}$ is the certainty-equivalent of future utility, $\mathcal{V}_{i,t} = \Psi^{-1}\left(\mathbb{E}_{i,t}\left[\Psi\left(U^{-1}(V_{i,t+1})\right]\right)$.

The labor disutility coefficient is indexed by *lagged* productivity, $\xi_t = \xi A_{t-1}$ and acts as a long-run wealth effect—as in Jaimovich and Rebelo (2009), this ensures that hours are stationary. We adopt functional forms: $U(C) = \frac{C^{1-1/\psi}-1}{1-1/\psi}$ and $\Psi(Z) = \frac{Z^{1-\gamma}-1}{1-\gamma}$, where γ controls the relative risk aversion and ψ the elasticity of intertemporal substitution (EIS).[6] Since $U(\cdot)$ is only defined over positive values, *net consumption*, $C_{i,t} - \xi_t\frac{h^{1+\nu}_{i,t}}{1+\nu}$, must be positive.

Household i has beliefs $\{p^i_{ss'}\}$ regarding TFP growth x_{t+1} from state s to s' and forms an expectation $\mathbb{E}_{i,t}$ accordingly. Households are dogmatic, as in Chen et al. (2012) and Simsek (2013b): they *agree to disagree* and do not learn from the views of others (like in the polarized bipartisan politics of current times). Beliefs about productivity translate into beliefs about earnings and asset prices. Their differences are settled through financial trades.

Household i chooses consumption $C_{i,t}$, hours $h_{i,t}$, firm shares $S_{i,t}$, and risk-free bonds $B_{i,t}$ to maximize (3) subject to a flow budget constraint

$$C_{i,t} + Q_t S_{i,t} + B_{i,t} = R_{e,t}Q_{t-1}S_{i,t-1} + R_{b,t}B_{i,t-1} + W_t h_{i,t}. \tag{4}$$

[6] CRRA preferences correspond to the assumption $\psi = \gamma^{-1}$. Given the endogenous labor supply, γ controls but does not coincide with the risk aversion for lotteries on financial wealth (see e.g. Swanson 2018).

We denote the household's human wealth by:

$$\mathcal{H}_{i,t} = \mathbb{E}_t \left[\sum_{k=1}^{\infty} \Lambda_{t,t+k} \left(W_{t+k} h_{i,t+k} - \xi_{t+k} \frac{h_{i,t+k}^{1+\nu}}{1+\nu} \right) \right]. \tag{5}$$

Human wealth is the present discounted value of future *net labor income*. Net labor income equals the difference between labor earning minus labor disutility. The present value is discounted using the SDF $\Lambda_{t,t+k} = \prod_{j=1}^{k} \Lambda_{t+j-1,t+j}$. The SDF is the same as the one used to value firms.

Households face a natural borrowing limit $R_{e,t} Q_{t-1} S_{i,t-1} + R_{b,t} B_{i,t-1} + W_t h_{i,t} - \xi_t \frac{h_{i,t}^{1+\nu}}{1+\nu} \geq -\mathcal{H}_{i,t}$, given $S_{i,-1}$ and $B_{i,-1}$, where $R_{b,t}$ denotes the return on the risk-free bond and $R_{e,t} = \frac{Q_t + \pi_t}{Q_{t-1}}$ the return on equity. This borrowing limit corresponds to the maximum households can borrow without violating the non-negativity of their consumption net of labor disutility.

SDF and Equilibrium. The SDF can be inferred from the process from asset returns through no-arbitrage conditions:

$$1 = \mathbb{E}_t \left[\Lambda_{t,t+1} \frac{\pi_{t+1} + Q_{t+1}}{Q_t} \right], \qquad 1 = \mathbb{E}_t \left[\Lambda_{t,t+1} R_{b,t+1} \right]. \tag{6}$$

A competitive equilibrium is defined next.

Definition 1 (Competitive equilibrium)**.** *Given initial bond holdings and shares $\{B_{i,-1}, S_{i,-1}\}_{i=1}^{I}$ and hours h_0, a competitive equilibrium is a set of stochastic process for quantities $\{\{C_{i,t}, h_{i,t}, B_{i,t}, S_{i,t}\}_{i=1}^{I}, h_t\}$ and prices $\{W_t, R_{b,t}, Q_t\}$ such that*

(i) $\{h_{t+1}\}$ maximizes (2) given wages W_t and the SDF $\Lambda_{t,t+1}$.

(ii) $\{C_{i,t}, h_{i,t}, B_{i,t}, S_{i,t}\}$ maximizes (3) subject to (4) given prices, for $i \in \mathcal{I}$.

(iii) Markets for goods, labor, bonds, and shares clear

$$\sum_{i=1}^{I} \mu_i C_{i,t} = A_t h_t^{\alpha}, \qquad \sum_{i=1}^{I} \mu_i h_{i,t} = h_t, \qquad \sum_{i=1}^{I} \mu_i B_{i,t} = 0, \qquad \sum_{i=1}^{I} \mu_i S_{i,t} = 1. \tag{7}$$

3 Characterization

We now present a recursive representation of the Markov equilibrium in the exogenous state s and an aggregate endogenous state variable X, to be revealed below. The law of motion of X is given by a function χ to be solved for, i.e., $X' = \chi(X, s, s')$. All aggregate variables are functions of X and s, e.g. $R_{e,t+1} = R_e(X_t, s_t, s_{t+1})$.

We proceed as follows: First, we reduce the investor's problem, which originally includes labor supply and portfolio decisions, into a consumption-savings problem without labor. We then obtain explicit expressions for consumption and portfolio choices under the alternative representation. Finally, we recover labor, consumption, and asset prices in the original problem from the alternative representation.

3.1 Household's problem

The household problem is a portfolio problem with endogenous labor. In general, the combination of portfolio problems with labor choice complicates the rendering of closed-form expressions.[7] Thanks to complete-markets and GHH preferences, we show an *as-if result*: the original problem can be derived from a modified portfolio the problem without reference to labor decisions.

Toward that characterization, we first observe that under GHH preferences, there are no wealth effects. As a result, all households have the same first-order condition for their labor choice:

$$h_{i,t} = h_t \equiv \left(\frac{W_t}{\xi_t}\right)^{\frac{1}{\nu}}. \tag{8}$$

Because the labor supply is the same for all agents, so is their human capital. Thus from now on, we drop the agent subscript from their human capital, $\mathcal{H}_{i,t+1} = \mathcal{H}_{t+1}$.

Considering the optimal labor choice, we can define the return to human wealth:

$$R_{h,t+1} \equiv \frac{W_{t+1}h_{t+1} - \xi_{t+1}\frac{h_{t+1}^{1+\nu}}{1+\nu} + \mathcal{H}_{t+1}}{\mathcal{H}_t}.$$

That is, we treat human wealth as an asset where labor income minus its disutility, $W_{t+1}h_{t+1} - \xi_{t+1}h_{t+1}^{1+\nu}(1+\nu)^{-1}$ is its dividend. With the return to human wealth, we recast the households' flow budget constraint:

$$\tilde{C}_{i,t} + Q_t S_{i,t} + B_{i,t} + \mathcal{H}_t = R_{e,t}Q_{t-1}S_{i,t-1} + R_{b,t}B_{i,t-1} + R_{h,t}\mathcal{H}_{t-1} \equiv N_{i,t} \tag{9}$$

where $\tilde{C}_{i,t} \equiv C_{i,t} - \xi_t \frac{h_t^{1+\nu}}{1+\nu}$ defines *net consumption* and $N_{i,t}$ defines *total wealth*. The investor's total wealth, $N_{i,t}$, is the sum of financial and human wealth. Total wealth funds

[7] With homothetic preferences and no labor supply, the coefficient of relative risk aversion is independent of wealth. Labor adds 'background risk' that causes the pricing kernel to exhibit declining relative risk-aversion in financial wealth.

the terms in the left-hand side: current net consumption and the future holdings of stocks, bonds, and human wealth.

Observe that the household's objective is to maximize net consumption. Thus, from the modified budget constraint, (9), it is as if households hold portfolios of bonds and two risky assets: stocks and human wealth. However, because the returns to stocks and human wealth are perfectly correlated, only the exposure to risk in total wealth matters, regardless of the portfolio composition.

Given that the exposure to risk in total wealth is the only thing that matters for the household, it is natural to work with the sum of stocks and human wealth as the sole risky asset, which we call the *surplus claim*. The price of the surplus claim, is $A_{t-1}P_t$, where P_t is given by

$$P_t = \mathbb{E}_t \left[\sum_{k=0}^{\infty} \frac{\Lambda_{t,t+k}}{A_{t-1}} \left(A_{t+k} h_{t+k}^{\alpha} - \xi_{t+k} \frac{h_{t+k}^{1+\nu}}{1+\nu} \right) \right].$$

The dividend of the surplus claim is the social surplus: the sum of output minus labor disutility—both measured in terms of goods. We denote by $R_r(X,s,s')$ the return on the surplus claim. As we show formally in Appendix A.1, the households' problem can be written in terms of net consumption and the exposure to the surplus claim:

Problem 1 (modified household's problem)**.**

$$V_i(N,X,s) = \max_{\tilde{C}_i,\omega_i} (1-\beta) U\left(\tilde{C}_{i,t}\right) + \beta U\left(\mathcal{V}_i(N,X,s)\right), \tag{10}$$

where $\mathcal{V}_i(N,X,s) = \Psi^{-1}\left(\mathbb{E}_i\left[\Psi\left(U^{-1}(V_i(N',X',s'))\,|N,X,s\right]\right)$, $N' \geq 0$, *and subject to:*

$$N' = R_{i,n}(X,s,s')(N-\tilde{C}_i), \qquad R_{i,n}(X,s,s') = (1-\omega_i)R_b(X,s) + \omega_i R_r(X,s,s'). \tag{11}$$

In this problem, the household only chooses net consumption $\tilde{C}_i$ and his exposure to the surplus claim ω_i. We recover $\{C_{i,t}, h_{i,t}, S_{i,t}, B_{i,t}\}$ in the original problem through the following definitions:

$$W_t = \xi_t h_t^{\nu}, \quad C_{i,t} = \tilde{C}_{i,t} + \frac{\xi_t h_t^{1+\nu}}{1+\nu}, \quad Q_t S_{i,t} = \frac{\omega_{i,t}}{\omega_{e,t}}(N_{i,t} - \tilde{C}_{i,t}) - \frac{\omega_{h,t}}{\omega_{e,t}}\mathcal{H}_t, \quad B_{i,t} = N_{i,t} - \mathcal{H}_t - Q_t S_{i,t} - \tilde{C}_{i,t},$$

where $\omega_{k,t}$ satisfies $R_{k,t} = \omega_{k,t} R_{r,t} + (1-\omega_{k,t}) R_{b,t}$, for $k \in \{h_i, e\}$.

Complete markets are key to this result: with complete markets, there is a combination of bonds and the surplus claim that yields the same payoffs as human wealth. Thus, the investor's problem is akin to a problem where human wealth is traded and could be sold at time zero, as any other financial asset. Notice that although human wealth is constant across households, it influences their individual portfolios: The stock holdings, $Q_t S_{i,t}$, are given by the desire to expose total wealth to risk, $\omega_{i,t}(N_{i,t} - \tilde{C}_{i,t})$, minus the exposure inherent from labor income, $\omega_{h,t}\mathcal{H}_t$.

Solution to the modified household's problem. The solution to the modified household problem is as in models with homothetic preferences and linear budget sets: it admits aggregation and portfolio separation. The next lemma provides a characterization of the value function, the consumption function, and the Euler equations of investor i.

Lemma 1 (Consumption and Euler equations)**.** *The household's value function takes the form:*

$$V_i(N, X, s) = U\left(v_i(X, s)N\right), \tag{12}$$

where $v_i(X,s)$ denotes the wealth multiplier. The consumption-wealth ratio $c_i(X,s) = \frac{\tilde{C}_i(N,X,s)}{N}$ and Euler equations for investor $i \in \mathcal{I}$ are given by

(i) Consumption-wealth ratio.

$$c_i(X, s) = \frac{(\beta^{-1} - 1)^{\psi} \mathcal{R}_i(X, s)^{1-\psi}}{1 + (\beta^{-1} - 1)^{\psi} \mathcal{R}_i(X, s)^{1-\psi}}, \tag{13}$$

where $\mathcal{R}_i(X,s) \equiv \Psi^{-1}\left(\mathbb{E}_i\left[\Psi\left(v(X', s')R_{i,n}(X, s, s')\right) | X\right]\right)$.

(ii) Euler equation for an asset $j \in \{r, b\}$.

$$1 = \mathbb{E}_i\left[\Lambda_i(X, s, s')R_j(X, s, s')\right], \tag{14}$$

where, for $\theta \equiv \frac{1-\gamma}{1-\psi^{-1}}$, the investor's SDF is given by

$$\Lambda_i(X, s, s') = \beta^{\theta}\left(\frac{c_i(\chi(X, s, s'), s')N'}{c_i(X, s)N}\right)^{-\frac{\theta}{\psi}} R_{i,n}(X, s, s')^{-(1-\theta)}. \tag{15}$$

(iii) The wealth multipliers satisfy:

$$v_i(X, s) = U^{-1}\left[U\left(c_i(X, s)\right) + \beta U\left(\mathcal{R}_i(X, s)\left(1 - c_i(X, s)\right)\right)\right]. \tag{16}$$

The lemma shows that the value function equals the static utility evaluated at $v_i(X,s)N$. The wealth multiplier $v_i(X,s)$ is a measure of welfare: $v_i(X,s)N$ corresponds to the constant consumption level that achieves the investor's expected utility.[8]

The consumption-wealth ratio of investor i given in (13) coincides with the consumption-wealth ratio of a deterministic problem with return $\mathcal{R}_i(X, s)$. In turn, $\mathcal{R}_i(X, s)$ is a risk-adjusted portfolio return with states weighted by the multiplier $v(X', s')$. The effect of $\mathcal{R}_i$ on consumption depends only on the EIS coefficient, ψ, that captures substitution and income effects. When $\psi = 1$, income and substitution effects cancel out; the consumption-wealth ratio is $c_i(X, s) = 1 - \beta$. The following section derives several results under this parameterization.

The equations given by (14) correspond to the Euler equations that yield the portfolios weights ω_i. Since (14) holds for every agent, and there are an equal number of equations (for each household) as states, all agents value payoffs in different states by the same amount regardless of their beliefs.

Finally, equation (16) provides a recursion to obtain the wealth multiplier $v_i(X, s)$.

Expressing portfolios in terms of asset prices and beliefs. The no-arbitrage conditions (6) coincide with the Euler equations if we replace individual beliefs and discount factors by objective probabilities and the economy-wide SDF. Hence, the SDF of i can be recovered from:

$$\Lambda_i(X, s, s') = \frac{p_{ss'}}{p^i_{ss'}} \Lambda(X, s, s').$$

Thus, i's SDF is the economy-wide SDF, scaled by the ratio of objective to subjective probabilities. In turn, given the objective probabilities, $\Lambda(X, s, s')$ can be recovered from observed asset prices, by inverting the no-arbitrage conditions (6),

$$\Lambda(X, s, s') = \frac{1}{p_{ss'}} \frac{|R^e_r(X, s, -s')|}{\Delta R_r(X, s)}.$$

The SDF depends on the excess return of the risky asset, $R^e_r(X, s, s') \equiv R_r(X, s, s')/R_b(X, s) - 1$, relative to the difference in realized returns $\Delta R_r(X, s) \equiv R_r(X, s, H) - R_r(X, s, L)$, a measure of risk.[9]

So far we have observed that investors agree on the value of one unit of consumption state by state, despite of disagreeing on their likelihood. For this, investors must be exposed to risk differently. In particular, relatively optimistic investors must be disproportionately levered, consistent with the optimism-waves narratives.

[8]This term is analogous to Lucas (1987) measure of the welfare cost of business cycle—the constant level of consumption that achieves the same utility as in the original stochastic economy.

[9]Notice that $[R_r(X, s, s') - R_b(X, s)]/\Delta R_r(X, s) = [R_e(X, s, s') - R_b(X, s)]/\Delta R_e(X, s)$, so we can use stocks or the surplus claim interchangeably.

The next lemma characterizes households' risk exposure (their portfolio weight in the surplus claim) in terms of the economy-wide SDF, market prices, and their beliefs.

Lemma 2 (Portfolio share)**.** *The shares of total wealth invested in the risky asset are*

$$\omega_i(X,s) = \frac{1}{\Delta R_r(X,s)} \left[\frac{\tilde{p}_i(X,s,H)}{p_{s,H}\Lambda(X,s,H)} - \frac{\tilde{p}_i(X,s,L)}{p_{s,L}\Lambda(X,s,L)} \right], \tag{17}$$

where $\tilde{p}_i(X,s,s')$ *is*

$$\tilde{p}_i(X,s,s') = \frac{(p^i_{ss'})^{\frac{1}{\gamma}} \left[v_i(\chi(X,s,s'),s') | R^e_r(X,s,s') | \right]^{\frac{1}{\gamma}-1}}{\sum_{\tilde{s}' \in \{L,H\}} (p^i_{s\tilde{s}'})^{\frac{1}{\gamma}} \left[v_i(\chi(X,s,\tilde{s}'),\tilde{s}') | R^e_r(X,s,\tilde{s}') | \right]^{\frac{1}{\gamma}-1}}.$$

See Appendix A.3. Lemma 2 describes how portfolio shares depend on the distorted probabilities $\tilde{p}_i(X,s,s')$ and $p_{ss'} \times \Lambda(X,s,s')$. The portfolio share of household i is increasing in p^i_{sH}; relatively optimistic investors hold more of the risky asset.

The portfolio weight $\omega_i(X,s)$ given by (17), is close to the ubiquitous Merton formula (see Merton (1969)), which relates portfolios to the expectation and variance of excess returns. In particular, as shown in Appendix A.3, $\omega_i(X,s)$ is approximately given by:

$$\omega_i(X,s) = \frac{\mu_{i,r}(X,s)}{\sigma^2_{i,r}(X,s)} + \mathcal{O}(\epsilon). \tag{18}$$

where we use the following approximation: $\mathbb{E}_i[R^e_r(X,s,s')] = \mu_{i,r}(X,s)\epsilon$, $Var_i[R_r(X,s,s')] = \sigma^2_{i,r}(X,s)\epsilon$, and $R_b(X,s) = 1 + r_b(X,s)\epsilon$ and log utility.

The takeaway from the approximation is that, as the expected excess return $\mu_{i,r}(X,s)$ is increasing in p^i_{sH}, more optimistic investors hold larger positions in the risky surplus claim. Thus, heterogeneity in beliefs translates into heterogeneity in portfolio shares. As a result, agents will be differently exposed to risk, their wealth share will fluctuate and this process will feedback into the economy-wide stochastic discount factor.

3.2 Firm's problem

We turn to the firm's hiring decision. The first-order condition of the firm yields the labor demand h_{t+1}:

$$\mathbb{E}_t \left[\Lambda_{t,t+1} \left(\alpha A_{t+1} h^{\alpha-1}_{t+1} - W_{t+1} \right) \right] = 0 \Rightarrow \alpha \mathcal{L}_t h^{\alpha-1}_{t+1} = w_{t+1}, \tag{19}$$

where $w_{t+1} \equiv W_{t+1}/A_t$ denotes a TFP-detrended wage. We dub $\mathcal{L}_t$ *the labor demand factor* (LDF). The LDF is the risk-neutral expectation of productivity growth:

$$\mathcal{L}_t = \mathbb{E}_t \left[\frac{\Lambda_{t,t+1}}{\mathbb{E}_t[\Lambda_{t,t+1}]} x_{t+1} \right].$$

Note that $\frac{p_{ss'}\Lambda_{t,t+1}}{\mathbb{E}_t[\Lambda_{t,t+1}]}$ is the risk-neutral probability at time t, see e.g. Duffie (2010) for a discussion. The LDF is consistent with a zero average labor wedge under the risk-adjusted expectations.

Recall that at t, the firm chooses future labor, h_{t+1}. Hence, current labor h_t depends on the previous period's LDF. Hence, $\mathcal{L}_{t-1}$ is an endogenous aggregate state variable with law of motion:

$$\mathcal{L}'(X,s) = \frac{p_{sL}\Lambda(X,s,L)}{p_{sL}\Lambda(X,s,L) + p_{sH}\Lambda(X,s,H)} x_L + \frac{p_{sH}\Lambda(X,s,H)}{p_{sL}\Lambda(X,s,L) + p_{sH}\Lambda(X,s,H)} x_H.$$

Expressing the labor demand factor in terms of asset prices. Let $\mathbb{E}_s[z_{s'}]$ denote the conditional mean and $\sigma_s[z_{s'}]$ denote the conditional volatility (standard deviation) of a variable $z_{s'}$ given s, using the objective probabilities $p_{ss'}$. The next proposition presents a convenient representation for the LDF:

Proposition 1. *The risk-neutral expectation of next period productivity growth is given by*

$$\mathcal{L}'(X,s) = \mathbb{E}_s[x_{s'}] - \underbrace{\frac{\mathbb{E}_s[R^e_r(X,s,s')]}{\sigma_s[R^e_r(X,s,s')]}}_{\textit{price of risk}} \underbrace{\sigma_s[x_{s'}]}_{\textit{quantity of risk}} . \tag{20}$$

where the Sharpe ratio is

$$\frac{\mathbb{E}_s[R^e_r(X,s,s')]}{\sigma_s[R^e_r(X,s,s')]} = \frac{\sigma_s[\Lambda(X,s,s')]}{\mathbb{E}_s[\Lambda(X,s,s')]}.$$

See Appendix A.4. Proposition 1 shows that the LDF depends on two components: First, a *quantity of risk*, the underlying risk in productivity growth, $\sigma_s[x_{s'}]$. Second, a *price of risk*, the required market compensation per quantity of risk. This price of risk is proportional to the volatility of the SDF. Thus, the LDF and the SDF are intimately connected.

Discussion: on the firm's objective. Given that labor is chosen in advance, hiring is risky. Hence, we must specify the SDF used by firms. This raises the question of what is an appropriate SDF. Under complete markets assumption, *any* pair of beliefs/SDF that correctly price stocks and bonds leads to the same firm value. In turn, the firm value is maximized using the firm's first-order condition under the economy-wide SDF. Every shareholder, regardless of their beliefs, agrees on what labor decision maximizes the firm wealth. While we assumed that firms compute expectations using the objective probabilities, any beliefs by managers would deliver the same labor market dynamics. How to determine the firm's objectives away from complete markets is still a matter for discussion in economics, see e.g. the discussion in Geanakoplos et al. (1990).

3.3 Labor Market Equilibrium

We combine the labor supply and demand schedules, equations (8) and (19), to obtain the labor market equilibrium:

$$h(\mathcal{L}) = \left(\frac{\alpha\mathcal{L}}{\xi}\right)^{\frac{1}{1+\nu-\alpha}}, \qquad w(\mathcal{L}) = \xi\left(\frac{\alpha\mathcal{L}}{\xi}\right)^{\frac{\nu}{1+\nu-\alpha}}. \tag{21}$$

Given that hours and wages are determined solely by the LDF, $\mathcal{L}$, realized profits also depend on the LDF and realized productivity growth: [10]

$$\pi(\mathcal{L}, s) = x_s\left(\frac{\alpha\mathcal{L}}{\xi}\right)^{\frac{\alpha}{1+\nu-\alpha}}\left[1 - \alpha\frac{\mathcal{L}}{x_s}\right].$$

Because labor is chosen in advance, profits may be negative, unless $x_L > \alpha x_H$.[11]

Discussion: Equity Premium and Labor Volatility Puzzles. The equilibrium in the labor market showcases that variations in the SDF provoke variations in the LDF, that is, they lead to labor fluctuations. This observation connects the equity premium, equity volatility, and labor volatility puzzles. Intuitively, when investors are more willing to bear risk, expected excess returns are low, and risk-adjusted probabilities put more weight on high TFP growth. As a result, the firm takes greater risks by hiring more labor. As we know from asset-pricing, from Hansen and Jagannathan (1991) and Cochrane and Hansen (1992), not only is the volatility of the SDF relatively large, but it is associated with substantial movements in expected returns (see e.g. Cochrane (2011)). Therefore, success in generating large employment fluctuations is tied to the success in obtaining a large and volatile equity premium because the labor volatility and equity volatility puzzles are the same puzzle in this setting.

[10] As hours and wages depend only on $\mathcal{L}$, we simplify notation and write $h(\mathcal{L})$ and $w(\mathcal{L})$ instead of the more general notation $h(X, s)$ and $w(X, s)$. Similarly, we write profits as $\pi(\mathcal{L}, s)$ instead of $\pi(X, s)$.

[11] Given that the highest possible value for $\mathcal{L}$, the last period expected TFP growth under the risk-neutral measure, is x_H, profits are positive as long as $x_s > \alpha\mathcal{L}$. The condition $x_L > \alpha x_H$ guarantees this is the case.

3.4 Markov equilibrium

In addition to $\mathcal{L}$, the *wealth distribution* is also an endogenous state variable. We define the share of (total) wealth of investor $i \in \mathcal{I}$:

$$\eta_{i,t} \equiv \frac{\mu_i N_{i,t}}{\sum_{j=1}^{I} \mu_j N_{j,t}}.$$

$\eta_{i,t}$ evolves according to

$$\eta_i'(X,s,s') = \frac{\eta_i R_{i,n}(X,s,s')(1-c_i(X,s))}{\sum_{j=1}^{I} \eta_j R_{j,n}(X,s,s')(1-c_j(X,s))}. \tag{22}$$

Given $\mathcal{L}$ and $\{\eta_i\}_{i=1}^{I-1}$, and the realization of TFP growth, we can characterize all aggregate variables. We stack the endogenous state variables in $X \equiv \left(\mathcal{L}, \{\eta_i\}_{i=1}^{I-1}\right)$ and define a Markov equilibrium in (X,s).

Definition 2 (Markov Equilibrium)**.** *A Markov equilibrium in (X,s), with a law of motion for X given by (20) and (22), is the set of functions: price of surplus claim $P(X,s)$, interest rate $R_b(X,s)$, labor hours $h(\mathcal{L})$, wages $w(\mathcal{L})$, wealth multiplier $v_i(X,s)$ and policy functions $(c_i(X,s), \omega_i(X,s))$, for $i \in \{1,\ldots,I\}$, such that:*

i. The value function is given by (12) and satisfies the Bellman equation (10). The consumption-wealth ratio is given by (13) and the portfolio share is given by (17).

ii. Hours and wages satisfy (21).

iii. The goods and the risky asset markets clear:

$$\sum_{i=1}^{I} \eta_i c_i(X,s) = \frac{x_s h(\mathcal{L})^{\alpha} - \xi \frac{h(\mathcal{L})^{1+\nu}}{1+\nu}}{P(X,s)}, \qquad \sum_{i=1}^{I} \tilde{\eta}_i(X,s)\omega_i(X,s) = 1, \tag{23}$$

where $\tilde{\eta}_i(X,s) \equiv \frac{\eta_i(1-c_i(X,s))}{\sum_{j=1}^{I}\eta_j(1-c_j(X,s))}$.

4 Analytic Solution: Log Utility Case

In this section, we consider log utility. We show that the LDF is a function of only market beliefs. We then derive a formula that dictates how market beliefs evolve. This analytic formulation renders sharp predictions about how belief heterogeneity amplifies or dampens business cycles. In the subsequent section, we supplement this characterization with a quantitative exercise.

Equilibrium Labor Demand Factor Given Market Beliefs. Under log utility, $\psi = \gamma = 1$. In this case, the consumption-wealth ratio and portfolio shares specialize to:

$$c_i(X) = 1 - \beta, \qquad \omega_i(X,s) = \frac{1}{\Delta R_r(X,s)} \left[\frac{p^i_{sH}}{p_{sH}\Lambda(X,s,H)} - \frac{p^i_{sL}}{p_{sL}\Lambda(X,s,L)} \right].$$

Combined with the market clearing conditions (23), they yield the risk-free rate and the risk premium.

Proposition 2 (Risk-free rate and risk premium)**.** *Let $\psi = \gamma = 1$. Then,*

(i) The risk-free rate is

$$R_b(X,s) = \left(1 - \frac{\alpha}{1+\nu}\right) \frac{x_s}{\beta} \frac{\mathcal{L}'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}}}{x_s \mathcal{L}^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu}\mathcal{L}^{\frac{1+\nu}{1+\nu-\alpha}}}. \tag{24}$$

$R_b(X,s)$ *is increasing in* $\mathcal{L}'(X,s)$ *and decreasing in* x_s*.*

(ii) The conditional risk premium is given by

$$\mathbb{E}_s[R^e_r(X,s,s')] = \frac{1}{1 - \frac{\alpha}{1+\nu}} \frac{\mathbb{E}_s[x_{s'}] - \mathcal{L}'(X,s)}{\mathcal{L}'(X,s)}. \tag{25}$$

$\mathbb{E}_s[R^e_r(X,s,s')]$ *is decreasing in* $\mathcal{L}'(X,s)$*.*

Proof. See Appendix A.5. □

Proposition 2 shows that the risk-free rate and the risk premium can be deduced from the current productivity growth, x_s, and the lagged and current period's LDF, $\mathcal{L}$ and $\mathcal{L}'(X,s)$. The risk premium and the return on safe assets move in opposite directions with labor demand conditions: an increase of $\mathcal{L}'(X,s)$ leads to a decline in the risk premium, but leads to an increase in the risk-free rate. Ceteris paribus, periods of low risk premium are associated with a high labor demand.

Of course, the LDF $\mathcal{L}'(X,s)$ is endogenous and a function of market beliefs. Next, we solve for $\mathcal{L}'(X,s)$ and, thus, determine asset prices. We start by translating the market clearing conditions into a demand and supply system, where the quantity variable is the risk in investors' portfolios. The demand and supply of risk are a convenient transformation of the asset-market clearing conditions (23): multiply both sides of the condition for risky assets by $\sigma_s[R_r(X,s,s')]$, and use that $\sigma_s[R_{i,n}(X,s,s')] = \omega_i(X,s)\sigma_s[R_r(X,s,s')]$, to obtain:

$$\underbrace{\sum_{i=1}^{I} \eta_i \sigma_s[R_{i,n}(X,s,s')]}_{\text{demand for risk}} = \underbrace{\sigma_s[R_r(X,s,s')]}_{\text{supply of risk}}.$$

The left of the expression is the *demand for risk*. The demand for risk corresponds to the volatility in total wealth that households are willing to accept at given asset prices. The right-hand side represents the *supply of risk*. The supply of risk is the volatility of the economy's surplus claim. The next proposition expresses the demand and supply of risk as a function of the LDF.

Proposition 3 (The demand and supply of risk)**.** *Suppose $x_L > \alpha x_H$. Then,*

(i) The supply of risk is

$$\sigma_s[R_r(X,s,s')] = \frac{x_s}{\beta} \frac{\sigma_s[x_{s'}]\mathcal{L}'(X,s)^{\frac{\alpha}{1+\nu-\alpha}}}{x_s \mathcal{L}^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu}\mathcal{L}^{\frac{1+\nu}{1+\nu-\alpha}}}. \tag{26}$$

(ii) The demand for risk is

$$\sum_{i=1}^{I} \eta_i \sigma_s[R_{i,n}(X,s,s')] = \sigma_s[x_{s'}]R_b(X,s)\left[\frac{\overline{p}^m_{sH}(X)}{\mathcal{L}'(X,s) - x_L} - \frac{\overline{p}^m_{sL}(X)}{x_H - \mathcal{L}'(X,s)}\right], \tag{27}$$

where $\overline{p}^m_{ss'}(X) \equiv \sum_{i=1}^{I} \eta_i p^i_{ss'}$.

Proof. See Appendix A.6. □

Proposition 3 implies that the supply of risk is increasing in $\mathcal{L}'(X,s)$, while the demand for risk is decreasing in $\mathcal{L}'(X,s)$, as shown in Figure 1. As with any demand system, the equilibrium $\mathcal{L}'(X,s)$ falls in the intersection.

Let's delve into the supply of risk. The risk of the surplus claim increases in $\mathcal{L}'(X,s)$ through an *operating leverage channel*.[12] Recall that labor demand increases with $\mathcal{L}'(X,s)$. Because labor is hired in advance, costs are fixed while revenues are risky. Thus, there is

[12]Operating leverage is the ratio of revenues minus contemporary variable costs (which are zero in our setting) to profits. The relation between return risk and operating leverage originally appears in Lev (1974). For evidence on this channel, see e.g. Novy-Marx (2010) and Donangelo et al. (2019).

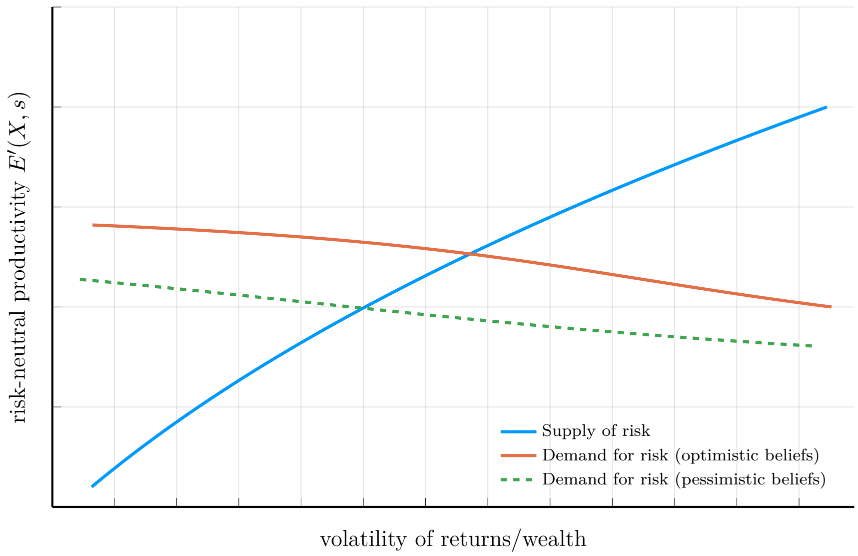


Figure 1: Equilibrium in the market for risky assets

more risk with a greater LDF, $\mathcal{L}'(X,s)$, as the following formula demonstrates:

$$\frac{\sigma_s[R_r(X,s,s')]}{\mathbb{E}_s[R_r(X,s,s')]} = \frac{\sigma_s[x']}{\mathbb{E}_s[x_{s'}]} \underbrace{\frac{\mathbb{E}_s[x_{s'}]\mathcal{L}'(X,s)^{\frac{\alpha}{1+\nu-\alpha}}}{\mathbb{E}_s[x_{s'}]\mathcal{L}'(X,s)^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu}\mathcal{L}'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}}}}_{\text{operating leverage}} > \frac{\sigma_s[x_{s'}]}{\mathbb{E}_s[x_{s'}]}. \tag{28}$$

In words, the risk in the aggregate surplus is an amplified version of the underlying risk, $\frac{\sigma_s[x_{s'}]}{\mathbb{E}_s[x_{s'}]}$, where the amplification factor is the operating leverage. Through this amplification, actual earnings are also more volatile than consumption, as in the data.[13]

Let's turn to the demand for risk. The demand for risk is decreasing in $\mathcal{L}'(X,S)$. As shown in Proposition 2, the risk premium is inversely related to $\mathcal{L}'(X,S)$, so investors are more willing to hold risky assets when the risk premium is high. The demand for risk is itself a function of *market beliefs*, a weighted average of investors' beliefs, $\overline{p}^m_{ss'}(X)$. As market beliefs become more optimistic, investors are willing to hold more risky assets for a given level of risk premium.

Propositions 2 and 3 demonstrate that, unlike models where hiring occurs after productivity is realized, labor demand and asset prices are determined jointly. We combine the demand and supply for risk to obtain the equilibrium LDF as a function of current market beliefs.

[13] If labor were chosen after productivity is known, dividends and consumption would be equally volatile. With preset labor, TFP shocks disproportionately impact dividends. A pattern of more volatile dividends than consumption is consistent with the data (see e.g. Campbell 2003), but, thus, inconsistent with models in which there is no risk in hiring.

Corollary 1 (Labor demand factor)**.** *The risk-neutral expectation of productivity growth $\mathcal{L}'(X,s)$ corresponds to the smallest real root of the quadratic equation:*

$$\frac{\alpha}{1+\nu}\mathcal{L}'(X,s)^2 - \left[\overline{p}^m_{sH}(X)\left(x_L + \frac{\alpha}{1+\nu}x_H\right) + \overline{p}^m_{sL}(X)\left(\frac{\alpha}{1+\nu}x_L + x_H\right)\right]\mathcal{L}'(X,s) + x_L x_H = 0.$$

Proof. See Appendix A.6. □

This corollary shows that, for the log case, the LDF is independent of the current state s and it is a function of market beliefs only. Hence, the wealth distribution affects asset prices and labor decisions to the extent it affects market beliefs. An important implication of this result is that there is no volatility in hours with common iid beliefs. As the productivity growth is likely close to iid under the objective measure, this result indicates the importance of non-rational beliefs in generating business cycle fluctuations.

Figure 1 also illustrates how we can exploit this demand system representation to explain the effects of changes in market beliefs. When market beliefs become pessimistic, there is a decline in the demand for risk, which leads to a decrease in the equilibrium LDF and, ultimately, a drop in hours.

From the LDF to the Evolution of Market Beliefs. Corollary 1 shows that, given market beliefs, we can compute the LDF and asset prices. Another convenient property of log preferences is that the law of motion of market beliefs, and wealth, has an analytic representation:

Proposition 4 (Dynamics of wealth and market beliefs)**.** *Let $\psi = \gamma = 1$. Then,*

(i) the wealth share of investor $i \in \mathcal{I}$ evolves as:

$$\eta_i'(X,s,s') = \eta_i \frac{p^i_{ss'}}{\overline{p}^m_{ss'}(X)}, \tag{29}$$

(ii) market beliefs are:

$$\overline{p}^m_{s'H}(X') = \sum_{i=1}^{I} \eta_i \frac{p^i_{ss'}}{\overline{p}^m_{ss'}(X)} p^i_{s'H}. \tag{30}$$

Proof. See Appendix A.7. □

Proposition 4 shows that the wealth of household i increases when it assigns a greater likelihood to the realized state than what market beliefs do. Next, we present a taxonomy of belief types. The analytic representation in Corollary 1 and Proposition 4 provides a tool to make predictions regarding how different forms of beliefs impact business cycles.

4.1 A Taxonomy of Beliefs and Business Cycle Predictions

We classify belief structures and show how each belief structure has different business cycle implications. We provide the following taxonomy:

Definition 3 (Taxonomy of beliefs)**.** *Household i is optimistic (pessimistic) relative to a benchmark belief o at state s if $p^i_{sH} > p^o_{sH}$ ($p^i_{sH} > p^o_{sH}$). We further classify belief structures:*

*(i) Beliefs are **rank preserving** if i is optimistic or pessimistic relative to o, $\forall s$.*

*(ii) Beliefs are **rank alternating** if i is optimistic relative to o in one state, but i is pessimistic relative to o in the other state.*

In this definition, the index o can represent another household's beliefs, the market's beliefs, or a rational expectations benchmark. We exploit this taxonomy to show that different belief structures induce different amplification properties, both in terms of the amplitude and phase of the business cycle. First, we investigate the different predictions under homogeneous non-rational beliefs. We then discuss how heterogeneous non-rational beliefs produces internal propagation mechanism and discuss the different properties of different belief structures.

Homogeneous beliefs. With homogeneous beliefs, we have a representative investor. In this case, the model lacks internal propagation: the LDF only depends on the current state, x_s. Furthermore, if investors believe that productivity growth is iid, $p_{LH}(X) = p_{HH}(X)$, then the LDF and, thus, hours are constant.

Away from iid growth, beliefs may amplify or dampen cycles relative to rational expectations. Figure 2 helps illustrate this point. The figure simulates four periods of recession within a twenty period interval for five types of homogeneous belief classifications: rational expectations ($p^i_{ss'} = p_{ss'}$ for all s, s'); two rank-preserving cases: always optimistic ($p^i_{sH} > p_{sH}$ for all s) or always pessimistic ($p^i_{sH} < p_{sH}$ for all s); and two rank-alternating cases: extrapolative ($p^i_{ss'} > p_{ss'}$ all $s = s'$) defined as optimistic during booms but pessimistic at busts, and intrapolative ($p^i_{ss'} < p_{ss'}$ all $s = s'$) defined as pessimistic at booms but pessimistic at busts.

Among rank-preserving beliefs: if beliefs are optimistic relative to rational expectations, the cycle is amplified at expansions, but dampened during recessions. Analogously, recessions are amplified and expansions dampened for pessimistic beliefs. Thus, rank-preserving beliefs have state-dependent amplification properties and, unconditionally, are ambiguous about amplification. Among rank-alternating beliefs: Extrapolative

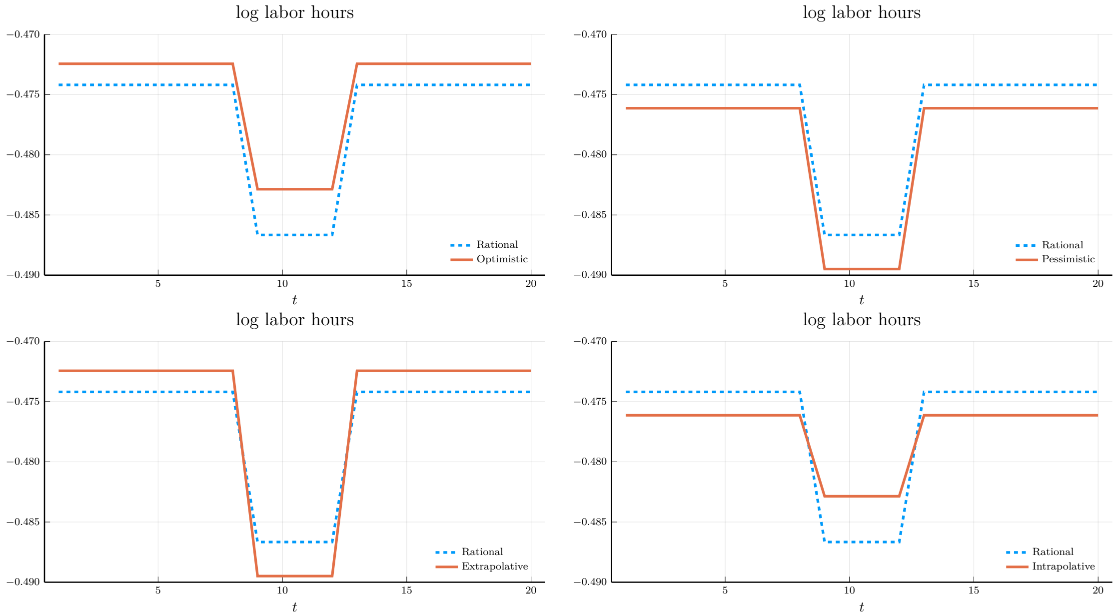


Figure 2: Homogeneous beliefs

Note: examples of cycles (four periods of bad shocks with sixteen periods of expansions) with homogeneous beliefs: rational, always optimistic (top-left panel), always pessimistic (top-right panel), extrapolative (bottom-left panel) and intrapolative (bottom-right panel) beliefs.

beliefs alternate, being optimistic at good states, but pessimistic at bad states. Under extrapolative beliefs, the amplitude of the cycle is magnified in all states. Analoguously, intrapolative beliefs produce the exact opposite and reduce the amplitude of the cycle in all states. In conclusion, the only class of belief structures that amplify the cycle relative to rational expectations in all states are extrapolative beliefs.

Next, we investigate the internal propagation induced by belief heterogeneity.

Heterogeneous beliefs. From Proposition 4, we deduce that $\eta_i'(X, s, H) > \eta_i$ if and only if investor i is optimistic at (X, s) relative to market beliefs. We know that the wealth of optimists increases after good shocks and decreases after bad shocks. This observation is key to the internal propagation of the economy:

Corollary 2. *If beliefs are heterogeneous, then:*

- *As the current state persists: the LDF increases (decreases) with time if the current state is high (low) growth.*
- *Consider an initial state (X, H) and a first switch from $s = H$ to $s' = L$ at some future date. Then,*

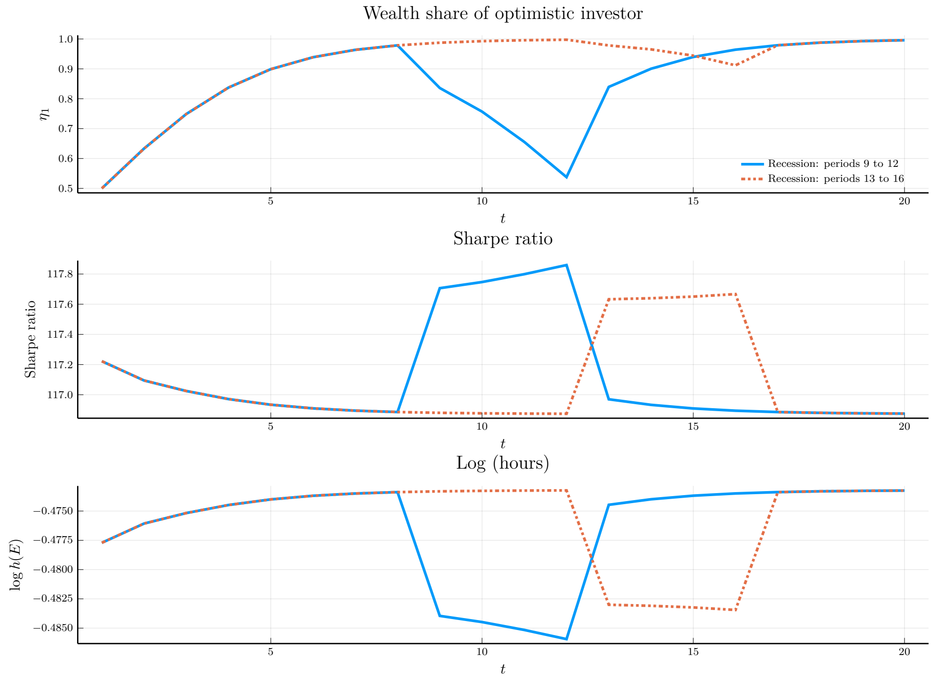


Figure 3: Heterogeneous beliefs: optimistic vs pessimistic

Note: examples of cycles (four periods of bad shocks with sixteen periods of expansions) with heterogeneous beliefs: $I = 2$ investors, always optimistic and always pessimistic. The top panel shows the wealth share of the optimistic investor, the middle panel shows the Sharpe ratio (under objective beliefs), and the bottom panel shows log labor hours.

- *If beliefs are rank-preserving, the later the date of the switch in state, the lower the reduction in output (the longer the boom the lesser the bust).*
- *If beliefs are rank-alternating, the later the date of the switch in state (the longer the boom), the lower the reduction in output (the longer the boom the greater the bust).*

Proof. See Appendix A.8. □

With belief heterogeneity, the economy evolves even without changes in the state. The reason is the joint evolution of wealth and beliefs. From Proposition 4, the wealth share of optimists increases while the economy stays in the boom phase. Thus, market beliefs become more optimistic relative to rational expectations as optimists accumulate wealth. This ultimately leads to an increase in the labor demand throughout the boom phase. The converse is true during low growth phase.

The connection between the length of cycles and their amplitude (the drop in output after a change in state) crucially hinges on whether beliefs are rank-preserving or rank-alternating. Rank-preserving beliefs attenuate the subsequent decline in TFP growth in a recession that follows a longer boom. The relationship between the duration of the high-growth phase and the severity of the recession is reversed with rank-alternating beliefs.

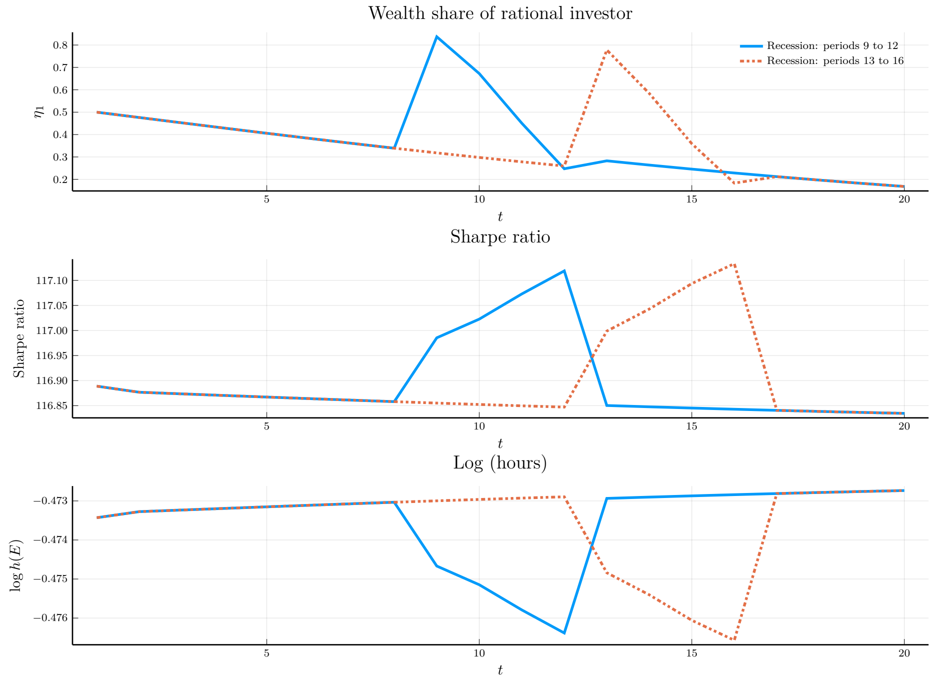


Figure 4: Heterogeneous beliefs: rational vs extrapolative

Note: examples of cycles (four periods of bad shocks with sixteen periods of expansions) with heterogeneous beliefs: $I = 2$ investors, rational and extrapolative. The top panel shows the wealth share of the optimistic investor, the middle panel shows the Sharpe ratio (under objective beliefs), and the bottom panel shows log labor hours.

Figure 3 aids us to explain this pattern. The figure shows simulations of business cycles in the case of rank-preserving beliefs—the figure uses $I = 2$, where investor 1 is optimistic and investor 2 is pessimistic in both states. The figure shows the evolution of the wealth share of the optimist, the Sharpe ratio, and log hours. The lower panel shows that the drop in hours is smaller as the economy remains longer in the high state. The intuition is that optimists accumulate wealth during the high-growth phase. This larger wealth implies that, even though they lose wealth, optimists arrive at the bad state with more wealth as the boom persists longer. Since optimists are optimist at all states, when beliefs are rank preserving, their greater wealth makes market beliefs also more optimistic during crashes. This attenuates the increase in risk premia and the decline in hours.

Figure 4, is the analogue figure for rank-alternating beliefs, where investor 1 is rational and investor 2 is extrapolative. The top panel shows that the wealth share of the rational investor declines during the boom phase. This means again that extrapolative investors are getting wealthier as the boom lasts longer. Consequently, extrapolative households have a larger wealth share during busts, precisely when they become pessimistic. As a result, market beliefs are also more pessimistic after the economy transition to the low state. This amplifies the increase in risk-premia and the decline in hours.

Frothy markets and risk build-ups. An implication of Corollary 2 is that the risk premium declines as booms last longer. To the extent that credit spreads are driven by risk premia, the model's prediction is also consistent with the discussions in López-Salido et al. (2017) and Krishnamurthy and Muir (2017) that argue that economic booms are characterized by "froth" market conditions driven by credit-market *sentiments*. Moreover, an increase in optimism leads not only to a reduction in risk premia, but also an increase in volatility, as shown in Figure 1. Under this interpretation, there is endogenous *risk build-up* during booms. As argued by Krishnamurthy and Li (2020), the combination of risk build-up and low spreads is challenging to generate for standard macro-finance models, such as He and Krishnamurthy (2013) and Brunnermeier and Sannikov (2014a). This observation suggests that market beliefs may be useful to explain credit and asset-market dynamics during boom-bust cycles.

Discussion: heterogeneity in beliefs vs. risk aversion. As discussed above, rank-alternating beliefs amplify inherent economic fluctuations. Therefore, it is not only differences in investors' risk appetite that matters for business cycles, but also how differences in the propensity to take risk to react to shocks. For instance, models of heterogeneous risk aversion are able to generate dispersion in portfolios and time variation in expected returns, but the ranking of agents in terms of risk-taking is always the same. This property is analogous to the case of rank-preserving beliefs shown in Figure 3. In contrast, there is no counterpart to the rank-alternating beliefs with heterogeneous risk aversion, as the least risk-averse agent in the boom is also the least risk-averse agent in the bust. Thus, different forms of investor heterogeneity have different implications for business cycle fluctuations, asset prices, and trading.

4.2 Trading volume

We now consider the implications for stock turnover, a measure of trading volume. To compute the volume of stock trading, we need first to map the portfolio holdings in terms of the surplus claim, ω_i, into the effective number of shares on firm equity in the primitive economy. This mapping is particularly simple in the case of linear labor disutility, $\nu = 0$ because in this case, human wealth is zero.[14] To simplify the exhibition, we adopt this assumption for the rest of the section.

[14]The share of wealth invested in stocks is simply ω_i, given that human wealth is equal to zero, $\mathcal{H}_i = 0$, and $R_e(X, s, s') = R_r(X, s, s')$ under this assumption.

With linear labor disutility, the volume traded is

$$\tau_t = \frac{1}{2}\sum_{i=1}^{I} |\omega_{i,t}\eta_{i,t} - \omega_{i,t-1}\eta_{i,t-1}|.$$

Clearly, there is no volume without heterogeneity. With heterogeneity, the volume traded depends on the level of disagreement. To study the effect of an increase in belief dispersion on volume, we consider a small deviation from homogeneous beliefs. We express investor i's beliefs as follows

$$p^i_{ss'} = p^*_{ss'} + \delta^i_{ss'}\epsilon,$$

where $\delta^i_{sH} + \delta^i_{sL} = 0$ and ϵ is a scalar that controls the degree of belief heterogeneity. Again, for parsimony, we focus on the case where $p^*_{ss'} = \frac{1}{2}$, such that, in the absence of heterogeneity, beliefs are iid and symmetric—the proofs in the Appendix hold for the general common belief case.

With the parameterization of beliefs used in this section, Appendix B.2 shows that the portfolio share of investor i is:

$$\omega_i(X, s; \epsilon) = 1 + \kappa_\omega \left[p^i_{sH} - \overline{p}^m_{sH}(X) \right] + \mathcal{O}(\epsilon^2),$$

where κ_ω is a positive constant. This expression showcases how optimistic investors, for whom $p^i_{sH} > \overline{p}^m_{sH}(X)$, are levered in stocks.

Consider current and future states s and s'. The effect of a perturbation in ϵ on the trades of investor i is:

$$\Delta S_i(X, s, s'; \epsilon) = \underbrace{\Delta\eta_i(X, s, s')}_{\text{rebalancing effect}} + \underbrace{\Delta\omega_i(X, s, s')\eta_i}_{\text{change-in-beliefs effect}} + \mathcal{O}(\epsilon^2), \tag{31}$$

as the economy switches from state (X, s) to (X', s'), where

$$\Delta\eta_i(X, s, s') \equiv \eta_i \frac{p^i_{ss'} - \overline{p}^m_{ss'}(X)}{p^*_{ss'}}, \qquad \Delta\omega_i(X, s, s') \equiv \kappa_\omega \left[p^i_{s'H} - \overline{p}^m_{s'H}(X) - (p^i_{sH} - \overline{p}^m_{sH}(X)) \right].$$

Expression (31) reveals two effects. The *rebalancing effect* captures the extent to which investors trade after a change in the state in order to keep portfolio shares constant: investors who put more likelihood on the realized state relative to the market belief, increased (decreased) their wealth share. Thus, they must buy (sell) the risky asset when that state is realized, in order to keep the portfolio share constant. Of course, as the economy evolves from s to s', portfolio shares themselves change as beliefs are modified. The *change-in-beliefs effect* captures the trade that follows the change in portfolio shares as the state changes. The change-in-beliefs effect is equal to zero if $s = s'$, as individual beliefs are constant in this case.

In tandem, the rebalancing and change-in-beliefs effects determine the equilibrium turnover.

Proposition 5 (Turnover)**.** *The economy's turnover, as it switches from state* (X, s) *to state* (X', s')*, is given by*

$$\tau(X, s, s'; \epsilon) = \frac{1}{2} \sum_{i=1}^{I} \eta_i \left| \frac{p^i_{ss'} - \overline{p}^m_{ss'}(X)}{p^*_{ss'}} + \kappa_\omega \left[p^i_{s'H} - \overline{p}^m_{s'H}(X) - (p^i_{sH} - \overline{p}^m_{sH}(X)) \right] \right| + \mathcal{O}(\epsilon^2). \tag{32}$$

Proof. See Appendix A.8. □

Proposition 5 provides a characterization of turnover. When $s = s'$, the change-in-beliefs effect vanishes; turnover is driven solely by the rebalancing effect:

$$\tau(X, s, s'; \epsilon) = \frac{1}{2} \sum_{i=1}^{I} \eta_i \frac{|p^i_{ss'} - \overline{p}^m_{ss'}(X)|}{p^*_{ss'}} + \mathcal{O}(\epsilon^2).$$

Thus, when there is no change in the state of the economy, turnover is proportional to the average absolute deviation of beliefs. The formula is consistent with the evidence in Section 5.4, which shows that dispersion on subjective beliefs about cash flows is correlated with stock market turnover.

The change-in-beliefs effect emerges when the economy switches states, that is, when $s \neq s'$, . In general, this effect may either amplify or dampen the rebalancing effect, depending on the type of belief and the direction of change in the economy. For instance, suppose that $s = H$ and $s' = L$ and that investors have rank-alternating beliefs. Optimistic investors lose wealth as the economy switches to a bad state. The rebalancing effect implies that they need to sell some of the risky assets to maintain their portfolio shares once stocks lose value. These investors also become pessimists in downturns, so this leads them to sell even more stocks. Thus, the two effects go in the same direction, amplifying the impact on the turnover when the economy switches from high to low states. The two effects go in opposite directions when $s = L$ and $s' = H$. Pessimists become optimistic as the economy switches to the good state, which induces them to increase their portfolio share in stocks while the rebalancing effect dictates them to sell stocks once stocks appreciate in order to keep the portfolio balanced.

Connecting with the Turnover Evidence. It is convenient to express heterogeneity in beliefs, p^i_{ss}, in terms of heterogeneity in the perceived *persistence* of fundamentals. Assuming investors agree on the unconditional mean of x_t, $\overline{x}$, we can write $\mathbb{E}_{i,t}[x_{t+1}] - \overline{x} = \theta_i(x_t - \overline{x})$, where θ_i is a function of $p^i_{ss'}$. The following corollary shows heterogeneous beliefs lead to larger turnover rates as the economy from booms to recessions.

Corollary 3. *Suppose investors agree on the unconditional mean of x_t, i.e. $p^i_{LH}/p^i_{HL} = \overline{p}_H/\overline{p}_L$ and that the following condition is satisfied: $p^*_{ss'} = \overline{p}_H = \frac{1}{2}$. Turnover as the economy switches from s to s' is given by*

$$\tau(X, H, L; \epsilon) = \frac{\zeta(s, s')}{2} \sum_{i=1}^{I} \eta_i |\theta_i - \theta(X)| + \mathcal{O}(\epsilon^2), \tag{33}$$

where

$$\zeta(s, s') \equiv \begin{cases} \kappa_\omega + 1, & \text{if } s = H \text{ and } s' = L \\ |\kappa_\omega - 1|, & \text{if } s = L \text{ and } s' = H \\ 1, & \text{if } s = s' \end{cases}$$

The key message from Corollary 3 is that turnover increases in belief dispersion and, furthermore, that the effect is more pronounced during busts. Both predictions are in line with the evidence discussed in Section 5.4. The assumption of rank-alternating beliefs is important to obtain this asymmetric effect. If investors have rank-preserving beliefs, where they are equally optimistic or pessimistic in both states, so $\tilde{\delta}^i_{s'H} = \tilde{\delta}^i_{sH}$ even for $s' \neq s$, then the change-in-beliefs effect will be equal to zero and we would not obtain a stronger response of turnover to disagreement during bad times. Therefore, rank-alternating beliefs are key to capturing the dynamics of stock market turnover.

5 Quantitative analysis

We now consider the quantitative implications. We extend the baseline model from Section 2 along two dimensions. First, productivity growth x_t now takes values on an arbitrary number of states. This extension is necessary to capture the empirical properties of earnings growth in the data. Second, we introduce mortality risk, to induce a non-degenerate stationary wealth distribution which is necessary to capture the effect of heterogeneous beliefs on unconditional moments.

5.1 The model with an arbitrary number of states

We start by specifying a continuous Markov process for x_t, under both objective and subjective beliefs, and then adopt a discrete approximation. Let $\hat{x}_t \equiv \log x_t - \mathbb{E}[\log x_t]$ denote demeaned log aggregate productivity. We assume that, under the objective measure, $\hat{x}_t$ follows the process:

$$\hat{x}_{t+1} = \sigma_u u_{t+1} + w_{t+1}, \tag{34}$$

where u_t is i.i.d., $u_{t+1}|w_{t+1} \sim \mathcal{N}(0,1)$, w_{t+1} is N_w-state Markov chain with transition probability $Pr(w_{t+1} = w^j | w_t = w^i) = p^w_{ij}$, and $\mathbb{E}[w_{t+1}] = 0$. Therefore, productivity is subject to Gaussian and non-Gaussian shocks needed to produce a fat-tailed earnings distribution. We focus on the case where $\hat{x}_t$ is iid, so $p^w_{ij} = p^w_{kj}$, for $i \neq k$.

We allow beliefs to deviate from the process (34), and potentially over- or under-react to past information. Moreover, we also allow for *sentiment shocks*. We use the following specification for the subjective beliefs about $\hat{x}_t$:

$$\hat{x}_{t+1} = \mathbb{E}_{i,t}[\hat{x}_{t+1}] + \sigma_{i,u} u_{i,t+1} + w_{t+1} \tag{35}$$

$$\mathbb{E}_{i,t}[\hat{x}_{t+1}] = \theta_i \hat{x}_t + \sigma_v v_t, \tag{36}$$

where $v_t \sim \mathcal{N}(0,1)$ and v_t is independent of both $u_{i,t}$ and w_t.

In two important ways, investors' beliefs deviate from the objective process for $\hat{x}_{t+1}$. First, investors disagree about the persistence of states. Some investors believe that positive shocks persist, $\theta_i > 0$, while others believe that shocks revert, $\theta_i < 0$. Second, beliefs are subject to a common sentiment shock, v_t. We impose that the unconditional variance of $\hat{x}_t$ under subjective beliefs coincides with the actual unconditional variance, $Var_i[\hat{x}_t] = \sigma^2_x$. Relative to the objective measure, sentiment shocks cause a larger fraction of the variation of $\hat{x}$ to be attributed to movements in expectations. Sentiment shocks allow us to reproduce the observed volatility of expectations in the data. We further assume that investors agree on the transition probabilities of w_{t+1} and observe its current value. Thus, investors understand that earnings have fat tail. Hence, the only source of disagreement is the persistence of shocks θ_i.

Appendix O2.6 discusses the discretization of the process for $\hat{x}_t$. The discretization provides a state space with dimension S for x_t, so $x_t \in \mathcal{X} = \{x^1, x^2, \ldots, x^S\}$, and transition probabilities $\{p^i_{ss'}\}$, for $s, s' \in \mathcal{S} = \{1, 2, \ldots, S\}$. Our discretization implies that the grid $\mathcal{X}$ is the same for all investors, so they agree on the state s, but they disagree on the transition probabilities $p^i_{ss'}$.

Mortality risk. We now assume that investors die with probability κ. When an investor dies, she leaves her net worth to her offspring, which will be of type $i \in \mathcal{I}$ with probability μ_i, the mass of type-i investors in the population. Investors derive no utility from their bequests. Hence, as in Blanchard (1985), the discount factor β reflects both impatience and the probability of death.

Mortality risk alters the law of motion of the wealth share η_i

$$\eta_i'(X, s, s') = (1-\kappa) \frac{\eta_i R_{i,n}(X, s, s')(1 - c_i(X, s))}{\sum_{j=1}^{I} \eta_j R_{j,n}(X, s, s')(1 - c_j(X, s))} + \kappa \mu_i.$$

As in Gârleanu and Panageas (2015), $\kappa > 0$ ensures that no investor type concentrates all the wealth asymptotically.

Model solution. We describe the model characterization with an arbitrary number of states, mortality risk, and general Epstein-Zin preferences in Appendix O1. Previous results are essentially unchanged.[15]

As in the binary case, an exact closed-form solution is unavailable away from the log case. The approximations used e.g. by Bansal and Yaron (2004) and Hansen et al. (2008), who considered economies with Epstein-Zin investors and time-varying expected growth rates, are not suitable for an economy with heterogeneity and non-Gaussian shocks. Therefore, we develop an alternative perturbation method that allows us to derive asymptotic closed-form expressions, described in detail in Appendix O1. An advantage of this method is that it is possible to compute the model solution even with a relatively large number of agents, which will be important to capture the heterogeneity of beliefs in the data.

5.2 Calibration

We use the following calibration, where parameter are expressed in quarterly terms. *Preferences*: We set $\beta = 0.99$, risk aversion $\gamma = 10.0$, and EIS $\psi = 2.0$, typical values in the literature. We choose the labor disutility parameter ξ to normalize the average hours to 1 and $\nu = 2$, which gives a Frisch elasticity of 0.5, in line with the micro estimates of Chetty et al. (2013). We set $\kappa = 0.02$, following Gârleanu and Panageas (2015). *Technology:* We set $\alpha = 0.66$ and choose $\mathbb{E}[\log x]$ and σ_u to match the average and standard deviation of annual consumption growth of 2% and 3.3%, respectively, consistent with Campbell and Cochrane (1999). The non-Gaussian states are two, $N_w = 2$, corresponding to *normal*

[15] For instance, Lemma 1 and Proposition 1 hold exactly, with $s, s' \in \{1, \ldots, S\}$.

times and *crisis periods*. We estimate the magnitude of the shock conditional on a crisis as follows. Using the time series for earnings and dividend growth in Robert Shiller's database, we compute the standard deviation of each series, Winsorized at 2.5% for each tail to capture the volatility in normal times. We then compute the average decline in growth for each series conditional on being in the left tail, i.e below the 2.5%-percentile. We find a drop in earnings growth of 4.9 standard deviations and a drop in dividend growth of 3.5 standard deviations. We set the drop in productivity to be 4.2 standard deviations in the crisis state, the average of these two numbers, and the probability of the crisis state to 2.5%.

Heterogeneity in investors' beliefs. The novelty of the paper is the estimation of the beliefs regarding the persistence of type-i investors, θ_i, the mass of type-i investors in the economy, μ_i, and their distribution, and the volatility of belief shocks σ_v.

To estimate these parameters, we exploit survey data on beliefs about earnings expectations. A key challenge is that the survey data regard individual firm expectations, so we infer the heterogeneity in beliefs about aggregate earnings from the cross-section.

We employ data from I/B/E/S on analysts' expectations about firms' future earnings. The I/B/E/S dataset presents quarterly analyst forecasts on earnings per share for several publicly traded firms. For each company, we may find one or more forecasts at different horizons.

We build firm-level series for the *expected standardized earnings growth*, $\mathbb{E}e_t^i$. We build these series through the following steps:

i. We compute total expected yearly earnings, by first multiplying EPS quarterly forecasts by the number of common outstanding shares. We then sum forecasts for the next 4 quarters to obtain the forecasted earnings over the next 12 months, following Gennaioli et al. (2016).

ii. We compute the difference between expected earnings in the next 4 quarters and *realized* earnings over the past 4 quarters.

iii. The difference in earnings depends on the size of the firm, hence we must standardize differences. Because some firms report negative realized earnings, we do not use earnings in the denominator to compute growth rates.[16] Instead, we standardize the change in earnings by the standard deviation of the firm's realized earnings during the sample.

[16] Gennaioli et al. (2016) exclude firms with negative earnings realizations. We chose a different standardization to avoid dropping observations with negative earnings.

We employ an econometric framework to estimate the parameters that govern beliefs as follows.

The econometric analysis is as follows. Let $i \in \mathcal{I}$ denote a firm-analyst observation in the I/B/E/S data—we index firm-level outcomes also by i. We denote (realized) earnings for i at period t by $e_{i,t}$ and its first-difference by $\Delta e_{i,t} = e_{i,t} - e_{i,t-1}$.[17] In turn, we denote aggregate earnings by e_t and the first-difference of aggregate earnings by Δe_t. We specify a process for the realized earnings of i in terms of exposure to the aggregate earnings:

$$\Delta e_{i,t} = \beta_i \Delta e_t + u_{i,t}, \tag{37}$$

where $u_{i,t} = \rho_i u_{i,t-1} + \epsilon_{i,t}$ and $\epsilon_{i,t} \sim \mathcal{N}(0, \sigma_\epsilon^2)$. The error term $\epsilon_{i,t}$ is i.i.d. and independent of Δe_t.[18] Idiosyncratic shocks are captured by $u_{i,t}$. In turn, ρ_i controls the persistence of idiosyncratic shocks. Hence, we assume that all firms are heterogeneous regarding their exposure to the aggregate earnings shock and the persistence of their idiosyncratic shocks.

To exploit the cross-section of the data, we focus on disagreement regarding the persistence of the shocks to aggregate earnings. For that, we assume that analysts agree that their stocks' earnings are exposed to the aggregate factor following (171) but disagree on the process for aggregated earnings. In particular, we assume that analyst i believes (in a dogmatic fashion) that Δe_t follows:

$$\Delta e_t = \theta_i \Delta e_{t-1} + \nu_{i,t}, \tag{38}$$

where $\nu_{i,t}$ is an i.i.d. process given by $\nu_{i,t} \sim \mathcal{N}(0, \sigma_\nu^2)$. Analysts agree on the unconditional mean for Δe_t, which we normalized to zero and Δe_t is perfectly observed. Thus, the expected change in aggregate earnings of analyst i is given by

$$\mathbb{E}_{i,t}[\Delta e_{t+1}] = \theta_i \Delta e_t, \tag{39}$$

where $\mathbb{E}_{i,t}[\cdot]$ denotes the conditional expectation at t of i. Hence, differences in beliefs are exclusively captured by differences regarding θ_i, the mean-reversion of aggregate earnings. For example, the formulation can capture belief extrapolation: a high value for θ_i implies that i is more optimistic about aggregate earnings after a positive shock and more

[17] As $e_{i,t}$ can potentially be negative, we work with first differences instead of proportional differences, $\frac{\Delta e_{i,t}}{e_{i,t}}$, or log-differences, $\Delta \log(e_{i,t})$. By focusing on the first differences, we do not have to drop firms that experience negative earnings, which is a significant fraction of our sample.

[18] For the exposition, we assume that $\Delta e_{i,t}$ and Δe_t have already been de-meaned, so we can omit the intercept, and that $\Delta e_{i,t}$ and Δe_t have been normalized to have unit variance.

pessimistic after a negative shock that an agent with a lower θ_j.

Expectations of changes in *individual* earnings depend on θ_i:

$$\mathbb{E}_{i,t}[\Delta e_{i,t+1}] = \beta_i \theta_i \Delta e_t + \rho_i u_{i,t}. \tag{40}$$

Equation (174) shows that we can infer properties of the unobserved process for subjective beliefs on *aggregate* earnings using the information on observed subjective beliefs about *individual* earnings.

We estimate $\{\beta_i, \rho_i, \theta_i\}_{i=1}^{I}$ using a multi-level Bayesian model. We describe the estimation procedure in detail in Appendix C.[19]

5.3 Quantitative Results: unconditional moments

We first study the model's ability to match unconditional moments. To isolate the role of each model ingredient, we start from a stripped-down version of the model and progressively add features until we reach the complete model. Table 1 presents the results.

The stripped-down version features a single representative rational investor and only Gaussian shocks. The first column shows the *endowment economy limit*. The economy behaves as an Epstein-Zin version of Mehra and Prescott (1985).[20] In this case, the volatility of consumption and dividends coincide, and the price-dividend ratio is constant. Column 2 introduces the hiring timing. Relative to the endowment economy limit, the surplus is more volatile than consumption and dividends are more volatile than the surplus. Quantitatively, the model matches the targeted volatility of consumption and delivers the volatility of dividends observed in the data which is not a target. However, iid beliefs do not produce excess stock volatility or variation in hours.

Column 3 adds the non-Gaussian shocks. In line with the literature on rare disasters, this raises the equity premium but does not add volatility.

Columns 4 and 5 consider the role of the non-rational *homogeneous* beliefs measured in the data. Column 4 assumes that investors have a common persistence parameter, $\theta_i = \theta$, set to the unconditional mean, but it abstracts from sentiment shocks, $\sigma_v = 0$. Relative to the case of rational beliefs, the model is now able to generate some volatility of hours. The sentiment shock adds substantial volatility to hours and equity returns.

Column 6 shows the results for the model with heterogeneous beliefs. This is key to

[19]The procedure is analogous to a ridge regression, where the estimates are regularized using an L2 penalty (see e.g. Hastie et al., 2009).

[20]We obtain this limit by making α and ξ go to zero, while α/ξ converges to a positive number. In this case, hours equal a positive constant, but both wages and labor disutility are equal to zero.

	Rational Beliefs (1)		Rational Beliefs (2)		Rational Beliefs (3)		Homogeneous Beliefs (4)		Homogeneous Beliefs (5)		Het. Beliefs (6)	
Variables	**Mean**	**Std**	**Mean**	**Std**	**Mean**	**Std**	**Mean**	**Std**	**Mean**	**Std**	**Mean**	**Std**
Interest rate	2.22	0.00	2.04	0.52	1.37	0.58	1.09	0.25	0.96	1.43	0.83	1.42
Excess Returns (surplus)	1.54	4.70	1.78	6.34	3.29	7.07	3.70	7.19	4.66	6.85	4.87	6.91
Excess Returns (equity)	1.55	4.70	1.79	6.35	3.31	7.08	3.80	8.61	4.71	10.80	5.01	11.43
Consumption growth	2.00	3.13	1.99	3.14	2.16	3.50	2.17	3.51	2.69	3.16	2.69	3.15
Surplus growth	2.00	3.13	1.99	4.36	2.24	4.85	2.24	4.86	2.70	4.37	2.69	4.37
Dividend growth	2.00	3.13	1.99	5.68	2.27	6.37	2.28	6.37	2.71	5.70	2.69	5.70
Log hours	0.00	0.00	-0.01	0.00	-0.51	0.00	-0.69	0.58	-0.72	2.16	-0.84	2.26

Table 1: Unconditional moments

deliver the turnover and leverage moments.

The model with all the features generates substantial labor market fluctuations, with a sizeable equity premium and equity volatility. Success in matching labor market outcomes is tightly connected to the improvement in asset-pricing moments. The excess volatility in stocks relative to the cash-flow volatility is a result of movements in beliefs. These, ultimately affect expected returns and hiring decisions.

5.4 Assessment of the Mechanism

In this section, we offer an assessment of the model's mechanism by investigating other data patterns. First, we show that periods with greater belief disagreement coincide with greater stock market turnover. Second, we present some suggestive evidence that investor expectations predict firm-level employment decisions. Finally, we discuss the performance in terms of asset-price predictability.

Belief heterogeneity and stock market turnover. A salient feature of models with heterogeneous beliefs is their predictions regarding stock-market turnover. In this section, we use our estimates to construct a time series of belief disagreements and present its correlation with a measure of stock turnover.

Recall that the expectation of analyst of aggregate earnings growth is given by

$\mathbb{E}_i[\Delta e_{t+1}] = \theta_i \Delta e_t$. We use the following *disagreement index DI_t,*:

$$DI_t = \underbrace{\overline{\sigma}[\theta_i] \times |\Delta e_t|}_{\overline{\sigma}[\mathbb{E}_i[\Delta e_{t+1}]]}. \tag{41}$$

The disagreement index (DI) has two components: First, the cross-sectional dispersion in θ_i. Second, the absolute value of current aggregate earnings growth, $|\Delta e_t|$. If all analysts were to agree on the persistence of aggregate earnings growth, such that $\overline{\sigma}[\theta_i] = 0$, then the DI would equal zero. In turn, given that Δe_t has already been demeaned, $|\Delta e_t|$ captures the distance of aggregate earnings growth to the mean. Also, there is no disagreement if aggregate earnings growth is at its average value, $|\Delta e_t| = 0$. Thus, the DI increases with the deviations from the mean of aggregate earnings and more so with the underlying disagreement about the reversal speed.

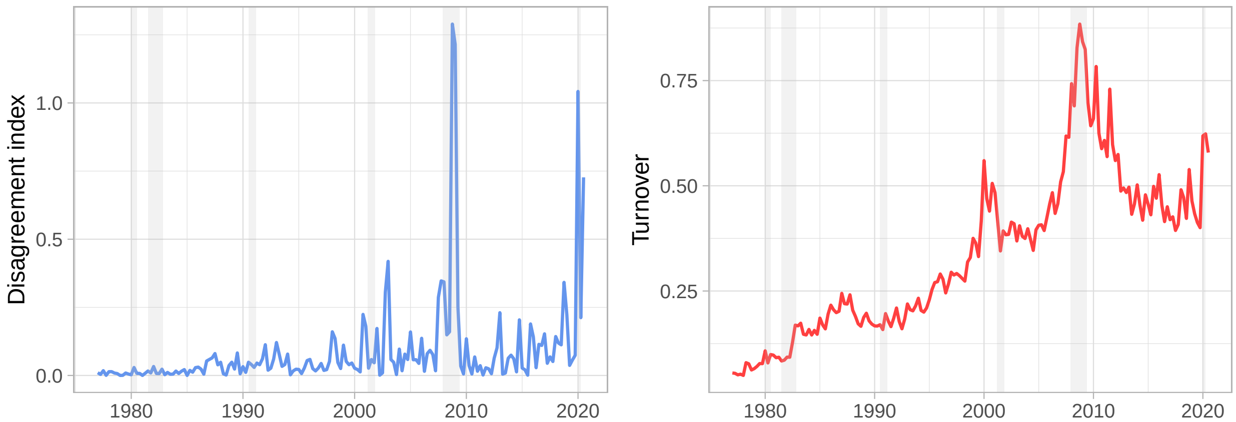
The left panel of Figure 7 shows the time series of DI. Disagreement is low during expansions, but spikes in recessions, the periods where earnings growth deviates most from the mean.

We relate the value-weighted stock-market turnover with the DI.[21] The right panel of Figure 7 shows that turnover increased significantly over time. Moreover, it shows an important countercyclical behavior.

Table 5 shows the result of a time-series regression of turnover against the DI. In line with the visual inspection of Figure 7, the disagreement DI features outliers during recessions. We thus exclude observations where the DI is below the 2.5% percentile or above the 97.5% percentile. Column (1) shows a strong statistically significant correlation between DI and turnover. If the disagreement index increases from the 25% percentile value to its 75% percentile value, turnover increases by almost 30%. Column (2) shows that turnover responds particularly strongly to the DI during downturns. We add an interaction of the DI with a dummy for NBER recessions and find that the coefficient is positive and statistically significant; its magnitude is economically large, and the response of turnover to the DI is 70% larger in recessions. Column (3) tests for a nonlinear relationship: we introduce a quadratic term, again excluding outliers. The quadratic term is non significant. Column (4) performs the same non-linear regression but includes outliers. In this case, the quadratic term becomes statistically significant, again indicating that the outliers also capture increased disagreement during downturns.

[21] We measure the stock turnover - shares traded divided by shares outstanding—for individual securities on the New York and American Stock Exchanges from January 1977 to December 2021. We measure turnover at the quarterly frequency and compute an aggregate turnover measure using a value-weighted average (similar results are obtained by using an equal-weight measure).

Figure 5: Time series of the disagreement index and stock market turnover



Note: Left panel shows the time series of the disagreement index and the right panel shows the time series of stock market turnover. The smooth line in the right panel is the HP-filter trend of turnover. The vertical bars represent NBER recessions.

In conclusion, the DI is strongly correlated with turnover, but more so in crises. Recall that above we show that extrapolative beliefs are consistent with this prediction.

Investor beliefs and firm employment decisions. In this section, we study if beliefs forecast firm-level labor decisions. We consider two measures of labor variables taken from Compustat Annual©: the realized growth in staff expenses (payroll) and the growth in the number of employees (workers).

We standardize measures of employment growth using the same method we use to standardize earnings. A key empirical link for the mechanism in the paper is to argue that investor expectations impact firm-level decisions, beyond lagged return.

The regression analysis is similar in spirit to the regression analysis in Gennaioli et al. (2016), which finds a positive correlation between firm-level capital investment and earnings expectations. Here, we regress the measures of realized employment growth against our firm-level data on earnings growth expectations. Following Gennaioli et al. (2016), we control for the firm's past 12-month stock returns and contemporaneous returns to control for information. We cluster standard errors at the firm level.

Table 3 reports that the earnings growth expectations predict the realized growth in the two employment measures, total wages, and the number of workers.

Asset-Price Predictability. (Campbell and Shiller, 1988), offer the following decompo-

Table 2: Regression of turnover on disagreement index

Dependent Variable:	*turnover*			
Model:	(1)	(2)	(3)	(4)
Variables				
(Intercept)	0.258***	0.262***	0.242***	0.255***
	(0.034)	(0.042)	(0.044)	(0.037)
DI	1.239***	1.066**	1.798**	1.260***
	(0.228)	(0.628)	(0.628)	(0.290)
$DI\times$ recession		0.742**		
		(0.253)		
DI^2			-2.068	-0.688**
			(1.692)	(0.209)
Fit statistics				
Observations	165	165	165	175
R^2	0.24084	0.26430	0.24786	0.30386
Adjusted R^2	0.23618	0.25522	0.23857	0.29576

Newey-West standard-errors in parentheses (4 lags)
*Signif. Codes: ***: 0.01, **: 0.05, *: 0.1*

Note: Columns (1) and (2) .

sition of the price dividend ratio:

$$pd_t = \kappa + \sum_{\tau=0}^{\infty} \rho^{\tau} \mathbb{E}_t^m [g_{t+\tau} - r_{t+\tau}], \tag{42}$$

where pd_t stands for the log price dividend ratio, g_t the growth of dividends and $r_{t+\tau}$ a subjective discount factor. Critically, $\mathbb{E}_t^m$ is the expectation of the representative market participant. This decomposition is key to understanding the role of heterogeneous beliefs.

Because market expectations $\mathbb{E}_t^m$ are unobservable, we approximate expectations $\mathbb{E}_t^s[g_{t+\tau}]$ where $\mathbb{E}_t^s$ is a statistical expectation constructed from econometric models. A challenge put forth by Cochrane (2008) is that since aggregate dividends are difficult to forecast with price dividend ratios, the bulk of the movement in stock values must be attributed to changes in discounting. At face value, this conclusion would seem to leave little room for earnings beliefs driving asset prices. However, using $\mathbb{E}_t^s$ to proxy for $\mathbb{E}_t^m$, assumes rational expectations.

Rather than assuming rational expectations, recent work has focused on approximating $\mathbb{E}_t^m$ directly from data on expectations surveys, $\mathbb{E}_t^i$. However, a concern regarding this

Table 3: I/B/E/S Expectations and Labor

Dependent Variable:	*Payroll*			Number of workers		
Model:	(1)	(2)	(3)	(4)	(5)	(6)
(Intercept)	0.388	0.274	0.364	0.392	0.143	0.170
	(0.247)	(0.223)	(0.241)	(0.097)	(0.195)	(0.052)
lag $\mathbb{E}^i[g_t]$	0.133***	0.135***	0.136***	0.061***	0.062***	0.067***
	(0.043)	(0.043)	(0.043)	(0.014)	(0.014)	(0.014)
lag 12-month return		0.474***			0.535***	
		(0.119)			(0.054)	
12-month return			0.234*			0.326***
			(0.125)			(0.043)
Fit statistics						
Observations	1797	1797	1797	1797	1797	1797
R2	0.081	0.090	0.083	0.081	0.090	0.083
Adjusted R2	0.038	0.046	0.039	0.038	0.046	0.039

Newey-West standard-errors in parentheses (4 lags)
*Signif. Codes: ***: 0.01, **: 0.05, *: 0.1*

approach is that surveys may not represent the beliefs of average investors.

We can recast the Campbell-Shiller decomposition in terms of $\mathbb{E}_t^s$, $\mathbb{E}_t^i$ and $\mathbb{E}_t^m$ to decompose beliefs into deviations from rational expectations and the representativeness of survey data.

$$pd_t = \kappa + \overbrace{\sum_{\tau=0}^{\infty} \rho^\tau \left[\mathbb{E}_t^m[g_{t+\tau}] - \mathbb{E}_t^i[g_{t+\tau}]\right]}^{\text{survey quality}} + \overbrace{\sum_{\tau=0}^{\infty} \rho^\tau \left[\mathbb{E}_t^i[g_{t+\tau}] - \mathbb{E}_t^s[g_{t+\tau}]\right]}^{\text{survey forecast}} \tag{43}$$

$$+ \overbrace{\sum_{\tau=0}^{\infty} \rho^\tau \left[\mathbb{E}_t^s[g_{t+\tau}] - \mathbb{E}_t^m[r_{t+\tau}]\right]}^{\text{rational expectations}}, \tag{44}$$

Under rational expectations, $\mathbb{E}_t^m = \mathbb{E}_t^s$, in which case beliefs do not impact substantially asset prices. If beliefs are irrational, $\mathbb{E}_t^m \neq \mathbb{E}_t^s$, leaving room for a substantial contribution of beliefs in the volatility of asset prices. In particular, this contribution will be captured by forecast errors.

We can use our model to construct counterfactual survey responses and deviations from rational expectations. Within our model, we find that earnings expectations can explain a large part of the fluctuations in the PE ratio, but not as if we take survey data as the measure of beliefs.

6 Conclusion

When asked about the nature of business cycles, Thomas Sargent[22], a pioneer of rational expectations, answered:[23]

> "[...] economists have been working hard to refine rational expectations theory. [...] An influential example of such work is the 1978 QJE paper by Harrison and Kreps. [...], for policymakers to know whether and how they can moderate bubbles, we need to have well-confirmed quantitative versions of such models up and running."

This response embraces the idea that "belief heterogeneity" matters but also calls for quantitative models that link beliefs with the real economy.

This paper responds to that call an adapts a standard real business cycle model to fit the narrative that waves of optimism and pessimism drive the business cycle. We spell out some key ingredients that are sufficient to account for that narrative. Namely, we argue that a combination of heterogeneity in beliefs, with substantial extrapolation, coupled with the assumption that labor is programmed before firms can observe shocks is needed to reproduce the data patterns.

With respect to the qualitative amplification of the business cycle, a few conclusions are worth emphasizing. First, as diagnostic investors accumulate wealth during booms and recessions, real business cycles are amplified. Second, the longer the boom period, the more severe the bust. Thirdly, the model is consistent with the countercyclical nature of turnover. We foresee that our framework can be extended to study the interaction with other forms of amplification, such as sticky prices or fire-sales externalities.

[22]Interestingly, Hyman Minsky was Thomas Sargent's undergraduate advisor. Whereas Sargent departs methodologically and calls for a quantitative approach to economic research, there is an agreement in the nature of business cycles.

[23]Interview with Thomas Sargent, *The Region*, August 26, 2010. Available at https://www.minneapolisfed.org/article/2010/interview-with-thomas-sargent.

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A Proofs

A.1 Derivation of the investor's modified problem

Proof. First, we adopt a change of variables and write the investor's problem as follows

$$V_{i,t} = \max_{\{\tilde{C}_{i,t}, h_{i,t}, B_{i,t}, S_{i,t}\}} (1-\beta)U\left(\tilde{C}_{i,t}\right) + \beta U\left(\Psi^{-1}\left(\mathbb{E}_{i,t}\left[\Psi\left(U^{-1}(V_{i,t+1})\right)\right]\right)\right), \tag{45}$$

subject to

$$\tilde{C}_{i,t} + Q_t S_{i,t} + B_{i,t} = R_{e,t} Q_{t-1} S_{i,t-1} + R_{b,t} B_{i,t-1} + W_t h_{i,t} - \xi_t \frac{h_{i,t}^{1+\nu}}{1+\nu}, \tag{46}$$

and the natural borrowing

$$(Q_t + \pi_t) S_{i,t-1} + R_{b,t} B_{i,t-1} + W_t h_{i,t} - \xi_t \frac{h_{i,t}^{1+\nu}}{1+\nu} \geq -\mathcal{H}_{i,t} \tag{47}$$

It is immediate that the optimal value of $h_{i,t}$ satisfies

$$W_t = \xi_t h_{i,t}^{\nu}. \tag{48}$$

We show next that, given $h_{i,t}$ satisfying (48), if the sequence $(\tilde{C}_{i,t}, B_{i,t}, S_{i,t})$ satisfies (46) and (47), then there exists $(N_{i,t}, \omega_{i,t})$ such that $(\tilde{C}_{i,t}, N_{i,t}, \omega_{i,t})$ satisfies (11) and $N_{i,t} \geq 0$. Conversely, if $(\tilde{C}_{i,t}, N_{i,t}, \omega_{i,t})$ satisfies (11) and $N_{i,t} \geq 0$, there exists $(B_{i,t}, S_{i,t})$ such that $(\tilde{C}_{i,t}, B_{i,t}, S_{i,t})$ satisfies (46) and (47).

From the definition of the return on human wealth, we have that $W_t h_{i,t} - \xi_t \frac{h_{i,t}^{1+\nu}}{1+\nu} = R_{h,t-1}\mathcal{H}_{i,t-1} - \mathcal{H}_{i,t}$, which allow us to write (46) and (47) as follows:

$$\tilde{C}_{i,t} + Q_t S_{i,t} + B_{i,t} + \mathcal{H}_{i,t} = N_{i,t}, \qquad N_{i,t} \geq 0. \tag{49}$$

We consider next the law of motion of total wealth:

$$N_{i,t+1} = \left[R_{e,t+1}\frac{Q_t S_{i,t}}{Q_t S_{i,t} + B_{i,t} + \mathcal{H}_{i,t}} + R_{b,t+1}\frac{B_{i,t}}{Q_t S_{i,t} + B_{i,t} + \mathcal{H}_{i,t}} + R_{h_i,t+1}\frac{\mathcal{H}_{i,t}}{Q_t S_{i,t} + B_{i,t} + \mathcal{H}_{i,t}}\right]\left(N_{i,t} - \tilde{C}_{i,t}\right). \tag{50}$$

As markets are dynamically complete, there exists replicating portfolios $(\omega_{h_i,t}, \omega_{e,t})$ such that

$$R_{k,t+1} = \omega_{k,t} R_{r,t+1} + (1-\omega_{k,t}) R_{b,t+1}, \tag{51}$$

for $k \in \{h_i, e\}$.

Combining the previous two conditions, we obtain

$$N_{i,t+1} = \left[R_{r,t+1} \frac{\omega_{e,t} Q_t S_{i,t} + \omega_{h_i,t} \mathcal{H}_{i,t}}{N_{i,t} - \tilde{C}_{i,t}} + R_{b,t+1} \frac{B_{i,t} + (1-\omega_{e,t}) Q_t S_{i,t} + (1-\omega_{h_i,t}) \mathcal{H}_{i,t}}{N_{i,t} - \tilde{C}_{i,t}} \right] \left(N_{i,t} - \tilde{C}_{i,t} \right). \tag{52}$$

Using the first condition in (49) to solve for $B_{i,t}$, we obtain

$$N_{i,t+1} = \left[(R_{r,t+1} - R_{b,t+1}) \, \omega_{i,t} + R_{b,t+1} \right] \left(N_{i,t} - \tilde{C}_{i,t} \right), \tag{53}$$

where $\omega_{i,t} \equiv \frac{\omega_{e,t} Q_t S_{i,t} + \omega_{h_i,t} \mathcal{H}_{i,t}}{N_{i,t} - \tilde{C}_{i,t}}$.

□

A.2 Proof of Lemma 1

Proof. First, we verify that the value function takes the form (12). Given the conjecture about the value function, the Bellman equation for investor i can be written as

$$\frac{(v_i(X,s)N)^{1-\psi^{-1}} - 1}{1-\psi^{-1}} = \max_{\tilde{c}_i, \omega_i} (1-\beta) \frac{(\tilde{c}_i N)^{1-\psi^{-1}} - 1}{1-\psi^{-1}} + \beta \frac{\mathbb{E}_i \left[(v_i(X',s')N')^{1-\gamma} \right]^{\frac{1-\psi^{-1}}{1-\gamma}} - 1}{1-\psi^{-1}}, \tag{54}$$

subject to $N' = R_{i,n}(X,s,s')(1-\tilde{c}_i)N$ and $N' \geq 0$.

The first-order conditions for the consumption-wealth ratio and the portfolio share are given by

$$(1-\beta)\tilde{c}_i^{-\psi^{-1}} = \beta \mathcal{R}_i(X,s)^{1-\psi^{-1}} (1-\tilde{c}_i)^{-\psi^{-1}} \tag{55}$$

$$0 = \mathbb{E}_i \left[(v_i(X',s') R_{i,n}(X,s,s'))^{-\gamma} v_i(X') (R_r(X,s,s') - R_b(X,s)) \right] \tag{56}$$

where $\mathcal{R}_i(X,s) = \mathbb{E}_i \left[(v_i(X',s') R_{i,n}(X,s,s'))^{1-\gamma} | X,s \right]^{\frac{1}{1-\gamma}}$.

Given $\mathcal{R}_i(X,s)$, we can solve for the consumption-wealth ratio:

$$\tilde{c}_i(X,s) = \frac{(\beta^{-1}-1)^{\psi} \mathcal{R}_i(X,s)^{1-\psi}}{1 + (\beta^{-1}-1)^{\psi} \mathcal{R}_i(X,s)^{1-\psi}}. \tag{57}$$

The envelope condition with respect to N is given by

$$v_i(X)^{1-1/\psi} = \beta \mathcal{R}_i(X)^{1-1/\psi} (1-\tilde{c}_i(X))^{-1/\psi} \Rightarrow \tilde{c}_i(X) = (1-\beta)^{\psi} v_i(X)^{1-\psi}. \tag{58}$$

From the optimality condition for the risky asset, we obtain

$$\mathbb{E}_i\left[(v_i(X',s')R_{i,n}(X,s,s'))^{1-\gamma}\right] = \mathbb{E}_i\left[v_i(X',s')^{1-\gamma}R_{i,n}(X,s,s')^{-\gamma}R_j(X,s,s')\right], \tag{59}$$

for $j \in \{r,b\}$.

Raising the envelope condition (58) to the power $\theta \equiv \frac{1-\gamma}{1-\psi^{-1}}$, using the definition of $\mathcal{R}_i(X)$ and condition (59), we obtain

$$1 = \mathbb{E}_i\left[\beta^\theta\left(\frac{v_i(X',s')}{v_i(X,s')}\right)^{1-\gamma}R_{i,n}(X,s,s')^{-\gamma}R_j(X,s,s')(1-\tilde{c}_i(X,s))^{-\theta/\psi}\right]. \tag{60}$$

Using the condition $v_i(X) = (1-\beta)^{\frac{1}{1-\psi^{-1}}}\tilde{c}_i(X)^{-\frac{\psi^{-1}}{1-\psi^{-1}}}$, we obtain the Euler equations

$$1 = \mathbb{E}_i\left[\beta^\theta\left(\frac{\tilde{c}_i(X',s')N'}{\tilde{c}_i(X,s)N}\right)^{-\frac{\theta}{\psi}}R_{i,n}(X,s,s')^{-(1-\theta)}R_j(X,s,s')\right]. \tag{61}$$

This concludes the derivation of the consumption-wealth ratio and the Euler equations for the two assets. It remains to check that the value function takes the form (12), which amounts to show that $v_i(X)$ indeed does not depend on N. Notice that $\tilde{c}_i(X,s)$ and $\omega_i(X,s)$ do not depend on N. We can then write the Bellman equation as follows:

$$v_i(X,s)^{1-\psi^{-1}} = (1-\beta)\tilde{c}_i(X,s)^{1-\psi^{-1}} + \beta\mathbb{E}_i\left[(v_i(X',s')R_{i,n}(X,s,s')(1-\tilde{c}_i(X)))^{1-\gamma}\right]^{\frac{1-\psi^{-1}}{1-\gamma}}, \tag{62}$$

for $\psi \neq 1$ and

$$\log v_i(X,s) = (1-\beta)\log\tilde{c}_i(X,s) + \beta\log\mathbb{E}_i\left[(v_i(X',s')R_{i,n}(X,s,s')(1-\tilde{c}_i(X,s)))^{1-\gamma}\right]^{\frac{1}{1-\gamma}}. \tag{63}$$

which is independent of N, which confirms our conjecture for the value function (12).

□

A.3 Proof of Lemma 2

Proof. The optimal portfolio share satisfies the condition

$$\frac{p^i_{sL}}{p^i_{sH}}\frac{v_i(\chi(X,s,L),L)^{1-\gamma}}{v_i(\chi(X,s,H),H)^{1-\gamma}}\left(\frac{\omega_i(X,s)R^e_r(X,s,L)+1}{\omega_i(X,s)R^e_r(X,s,H)+1}\right)^{-\gamma}\frac{|R^e_r(X,s,L)|}{R^e_r(X,s,H)} = 1 \tag{64}$$

Raising both sides to $-\frac{1}{\gamma}$, we obtain

$$\left(\frac{p^i_{sL}}{p^i_{sH}}\right)^{-\frac{1}{\gamma}} \frac{v_i(\chi(X,s,L),L)^{1-\frac{1}{\gamma}}}{v_i(\chi(X,s,H),H)^{1-\frac{1}{\gamma}}} \frac{\omega_i(X,s)R^e_r(X,s,L)+1}{\omega_i(X,s)R^e_r(X,s,H)+1} \frac{|R^e_r(X,s,L)|^{-\frac{1}{\gamma}}}{R^e_r(X,s,H)^{-\frac{1}{\gamma}}} = 1 \tag{65}$$

Rearranging the expression above, we obtain

$$\omega_i(X,s) = \frac{\tilde{p}_i(X,s,H)}{|R^e_r(X,s,L)|} - \frac{\tilde{p}_i(X,s,L)}{R^e_r(X,s,H)}, \tag{66}$$

where

$$\tilde{p}_i(X,s,s') = \frac{(p^i_{ss'})^{\frac{1}{\gamma}} \left[v_i(\chi(X,s,s'),s')|R^e_r(X,s,s')|\right]^{\frac{1}{\gamma}-1}}{\sum_{s' \in \{L,H\}} (p^i_{ss'})^{\frac{1}{\gamma}} \left[v_i(\chi(X,s,s'),s')|R^e_r(X,s,s')|\right]^{\frac{1}{\gamma}-1}}. \tag{67}$$

The SDF in this economy is given by

$$\Lambda(X,s,s') = \frac{1}{p_{ss'}} \frac{1}{R_b(X,s)} \frac{|R_r(X,s,-s') - R_b(X,s)|}{\Delta R_r(X,s)}, \tag{68}$$

where $\Delta R_r(X,s) = R_r(X,s,H) - R_r(X,s,L)$.

We can then write $\omega_i(X,s)$ as follows

$$\omega_i(X,s) = \frac{1}{\Delta R_r(X,s)} \left[\frac{\tilde{p}_i(X,s,H)}{p_{s,H}\Lambda(X,s,H)} - \frac{\tilde{p}_i(X,s,L)}{p_{s,L}\Lambda(X,s,L)}\right]. \tag{69}$$

Diffusion-like approximation. To better interpret the expression for the portfolio share, it is useful to consider an approximation analogous to the continuous-time limit for diffusion processes. Given $R_r(X,s,s')$, probabilities $p^i_{ss'}$ for household i, and a small parameter $\epsilon > 0$, we can find $\mu_{i,r}(X,s)$ and $\sigma_{i,r}(X,s)$ that satisfies the conditions

$$R^e_r(X,s,H) = \mu_{i,r}(X,s)\epsilon + \sqrt{\frac{p_{sL}}{p_{sH}}}\sigma_{i,r}(X,s)\sqrt{\epsilon}, \qquad R^e_r(X,s,L) = \mu_{i,r}(X,s)\epsilon - \sqrt{\frac{p_{sH}}{p_{sL}}}\sigma_{i,r}(X,s)\sqrt{\epsilon}, \tag{70}$$

which gives us the expected value and variance for household i:

$$\mathbb{E}_i[R^e_r(X,s,s')|X,s] = \mu_{i,r}(X,s)\epsilon, \qquad Var_i[R^e_r(X,s,s')|X,s] = \sigma^2_{i,r}(X,s)\epsilon. \tag{71}$$

Similarly, we can write $R_b(X,s) = 1 + r_b(X,s)\epsilon$.

From Equation (129), and assuming $\gamma = 1$, we obtain

$$\begin{aligned}\omega_i(X,s) &= R_b(X,s)\frac{p^i_{s,H}R^e_r(X,s,H) + p^i_{s,L}R^e_r(X,s,L)}{|R^e_r(X,s,L)|R^e_r(X,s,H)} \\ &= (1 + r_b(X,s)\epsilon)\frac{\mu_{i,r}(X,s)\epsilon}{\left(\sqrt{\frac{p_{sH}}{p_{sL}}}\sigma_{i,r}(X,s)\sqrt{\epsilon} - \mu_{i,r}(X,s)\epsilon\right)\left(\mu_{i,r}(X,s)\epsilon + \sqrt{\frac{p_{sL}}{p_{sH}}}\sigma_{i,r}(X,s)\sqrt{\epsilon}\right)},\end{aligned} \tag{72}$$

where we used the fact that $R^e_r(X,s,L) < 0$ by no-arbitrage.

In general, $(\mu_{i,r}(X,s), \sigma_{i,r}(X,s))$ and $p^i_{ss'}$ are functions of ϵ. Assuming that $\mu_{i,r}(X,s) = \mathcal{O}(1)$, $\sigma_{i,r}(X,s)) = \mathcal{O}(1)$, and $p^i_{ss'} = \mathcal{O}(1)$, we can write the expression $\omega_i(X,s)$ as follows:[24]

$$\omega_i(X,s) = \frac{\mu_{i,r}(X,s)}{\sigma^2_{i,r}(X,s)} + \mathcal{O}(\epsilon). \tag{73}$$

□

A.4 Proof of Proposition 1

Proof. First, we compute the Sharpe ratio on the risky asset. We will compute expectations using the objective measure, but a similar calculation gives the Sharpe ratio using the investors' subjective beliefs. The expected excess return is given by

$$\mathbb{E}\left[R^e_r(X,s,s')\right] = p_{sL}R^e_r(X,s,L) + p_{sH}R^e_r(X,s,H). \tag{74}$$

The variance of excess returns is given by

$$Var[R^e_r(X,s,s')] = p_{sL}p_{sH}\Delta R^e_r(X,s)^2. \tag{75}$$

The Sharpe ratio in the risky asset is then given by

$$\frac{\mathbb{E}[R^e_r(X,s,s')]}{\sqrt{Var[R^e_r(X,s,s')]}} = \sqrt{\frac{p_{sL}}{p_{sH}}}\frac{R^e_r(X,s,L)}{\Delta R^e_r(X,s)} + \sqrt{\frac{p_{sH}}{p_{sL}}}\frac{R^e_r(X,s,H)}{\Delta R^e_r(X,s)}. \tag{76}$$

We can write the expression above in terms of the economy's SDF. The SDF under the

[24] These assumptions are analogous to the ones used by e.g. Merton (1992) to derive the continuous-time limit with diffusion processes. Allowing for rare events, $p^i_{ss'} = \mathcal{O}(\epsilon)$ for some s', would lead to a jump-diffusion process.

objective measure can be written as

$$\Lambda(X,s,L) = \frac{\mathbb{E}[\Lambda(X,s,s')]}{p_{sL}} \frac{R_r^e(X,s,H)}{\Delta R_r^e(X,s)}, \qquad \Lambda(X,s,H) = -\frac{\mathbb{E}[\Lambda(X,s,s')]}{p_{sH}} \frac{R_r^e(X,s,L)}{\Delta R_r^e(X,s)}. \tag{77}$$

Combining the expressions above, we obtain

$$\frac{\mathbb{E}[R_r^e(X,s,s')]}{\sqrt{Var[R_r^e(X,s,s')]}} = \sqrt{p_{sL}p_{sH}} \frac{\Lambda(X,s,L) - \Lambda(X,s,H)}{\mathbb{E}[\Lambda(X,s,s')]}. \tag{78}$$

We consider next how the Sharpe ratio affects the risk-neutral expectation of future productivity growth. The risk-neutral expectation of productivity is given by

$$\mathbb{E}^Q[x_{t+1}] = p_{sL} \frac{\Lambda(X,s,L)}{\mathbb{E}[\Lambda(X,s,s')]} x_L + p_{sH} \frac{\Lambda(X,s,H)}{\mathbb{E}[\Lambda(X,s,s')]} x_H. \tag{79}$$

The difference between the expected value of productivity under the physical measure and the risk-neutral measure is given by

$$\mathbb{E}[x_{t+1}] - \mathbb{E}^Q[x_{t+1}] = p_{sL} \frac{\mathbb{E}[\Lambda(X,s,s')] - \Lambda(X,s,L)}{\mathbb{E}[\Lambda(X,s,s')]} x_L + p_{sH} \frac{\mathbb{E}[\Lambda(X,s,s')] - \Lambda(X,s,H)}{\mathbb{E}[\Lambda(X,s,s')]} x_H. \tag{80}$$

Rearranging the expression above, we obtain

$$\mathbb{E}[x_{t+1}] - \mathbb{E}^Q[x_{t+1}] = p_{sL}p_{sH} \frac{\Lambda(X,s,L) - \Lambda(X,s,H)}{\mathbb{E}[\Lambda(X,s,s')]} \Delta x, \tag{81}$$

where $\Delta x = x_H - x_L$.

Using the expression for the Sharpe ratio, we obtain

$$\mathbb{E}^Q[x_{t+1}] = \mathbb{E}[x_{t+1}] - \sqrt{p_{sL}p_{sH}} \frac{\mathbb{E}[R_r^e(X,s,s')]}{\sqrt{Var[R_r^e(X,s,s')]}} \Delta x. \tag{82}$$

□

A.5 Proof of Proposition 2

Proof. We start by deriving the process for returns. From the market clearing condition for goods, we obtain

$$\frac{x_s h(E)^\alpha - \xi \frac{h(E)^{1+\nu}}{1+\nu}}{P(X,s)} = 1 - \beta \tag{83}$$

The return on the surplus claim is given by

$$R_p(X,s,s') = \frac{x_s P(\chi(X,s,s'),s')}{P(X,s) - \left(x_s h(E)^\alpha - \xi \frac{h(E)^{1+\nu}}{1+\nu}\right)} = \frac{x_s}{\beta} \frac{x_{s'} h(E'(X,s))^\alpha - \xi \frac{h(E'(X,s))^{1+\nu}}{1+\nu}}{x_s h(E)^\alpha - \xi \frac{h(E)^{1+\nu}}{1+\nu}}. \quad (84)$$

Using the conditions in (21), we can rewrite the expression as follows

$$R_p(X,s,s') = \frac{x_s}{\beta} \frac{x_{s'} E'(X,s)^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu} E'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}}}{x_s E^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu} E^{\frac{1+\nu}{1+\nu-\alpha}}}. \quad (85)$$

Note that the denominator in the expression above is positive if and only if $E < \frac{1+\nu}{\alpha} x_s$. A sufficient condition is given by $\alpha x_H < x_L$, as shown below

$$E \leq x_H < \frac{x_L}{\alpha} < \frac{1+\nu}{\alpha} x_s, \quad (86)$$

and, similarly, this condition guarantees that the numerator is also positive.

Interest rate. The interest rate satisfies the condition $R_b(X,s) = \mathbb{E}\left[\frac{\Lambda(X,s,s')}{\mathbb{E}[\Lambda(X,s,s')]} R_p(X,s,s')\right]$, so $R_b(X,s)$ is given by

$$R_b(X,s) = \left(1 - \frac{\alpha}{1+\nu}\right) \frac{x_s}{\beta} \frac{E'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}}}{x_s E^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu} E^{\frac{1+\nu}{1+\nu-\alpha}}}, \quad (87)$$

using the fact that $\mathbb{E}\left[\frac{\Lambda(X,s,s')}{\mathbb{E}[\Lambda(X,s,s')]} x_{s'}\right] = E'(X,s)$.

The expression above is increasing in $E'(X,s)$, decreasing in x_s, and it is increasing in $\mathcal{L}$ for $s = L$.

Risk premium. The risk asset's excess return is given by

$$\frac{R_p(X,s,s')}{R_b(X,s)} = \frac{1}{1 - \frac{\alpha}{1+\nu}} \frac{x_{s'} - \frac{\alpha}{1+\nu} E'(X,s)}{E'(X,s)}. \quad (88)$$

The conditional risk premium is then given by

$$\mathbb{E}_s[R_p^e(X,s,s')] = \frac{1}{1 - \frac{\alpha}{1+\nu}} \frac{\mathbb{E}_s[x_{s'}] - E'(X,s)}{E'(X,s)}, \quad (89)$$

given the definition $R_p^e(X,s,s') \equiv \frac{R_p(X,s,s') - R_b(X,s)}{R_b(X,s)}$.

□

A.6 Proof of Proposition 3 and Corollary 1

Proof. We start by deriving the expression for the SDF. Note that we can express the SDF in terms of $R_p(X,s,s')$ and $R_b(X,s)$ instead of $R_r(X,s,s')$ and $R_b(X,s)$, as we can always obtain the SDF in terms of any two (linearly independent) assets. The SDF is then given by

$$\Lambda(X,s,s') = \frac{1}{p_{ss'}} \frac{1}{R_b(X,s)} \frac{|R_p^e(X,s,-s')|}{\Delta R_p^e(X,s)}. \tag{90}$$

The excess return on the surplus claim is given by

$$R_p^e(X,s,s') = \frac{x_s}{\beta} \frac{(x_{s'} - E'(X,s))E'(X,s)^{\frac{\alpha}{1+\nu-\alpha}}}{x_s E^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu} E^{\frac{1+\nu}{1+\nu-\alpha}}}. \tag{91}$$

Combining the previous two expressions, we obtain

$$\Lambda(X,s,s') = \frac{1}{p_{ss'}} \frac{1}{R_b(X,s)} \frac{|x_{-s'} - E'(X,s)|}{\Delta x} \tag{92}$$

using the fact that $\frac{R_p^e(X,s,s')}{\Delta R_p^e(X,s)} = \frac{x_{s'} - E'(X,s)}{\Delta x}$.

Demand for risk. The demand for risk in this economy is given by

$$\sum_{i=1}^{I} \eta_i \sigma_s[R_{i,n}(X,s,s')] = \sqrt{p_{sL}p_{sH}} \left[\frac{p_{sH}(X,s)}{p_{sH}\Lambda(X,s,H)} - \frac{p_{sL}(X,s)}{p_{sL}\Lambda(X,s,L)} \right], \tag{93}$$

where $p_{ss'}(X,s) = \sum_{i=1}^{I} \eta_{i,t} p_{ss'}^i$.

Using the expression for the SDF, the demand for risk can be written as

$$\sum_{i=1}^{I} \eta_i \sigma_s[R_{i,n}(X,s,s')] = \sigma_s[x_{s'}] \frac{\frac{1+\nu-\alpha}{1+\nu} \frac{x_s}{\beta}}{x_s E^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu} E^{\frac{1+\nu}{1+\nu-\alpha}}} \left[\frac{p_{sH}(X,s)E'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}}}{E'(X,s) - x_L} - \frac{p_{sL}(X,s)E'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}}}{x_H - E'(X,s)} \right], \tag{94}$$

given $\sigma_s[x_{s'}] = \sqrt{p_{sL}p_{sH}}\Delta x$.

The first term inside brackets in the expression above is decreasing in $E'(X,s)$ if and only if the following condition holds

$$\frac{1+\nu}{1+\nu-\alpha} E'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}-1}(E'(X,s) - x_L) - E'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}} < 0 \iff E'(X,s) < \frac{x_L}{\alpha}, \tag{95}$$

which holds, given that $E'(X,s) \leq x_H < \frac{x_L}{\alpha}$.

Therefore, the demand for risk is decreasing in $E'(X,s)$. As $E'(X,s)$ is decreasing in the Sharpe ratio of the risky asset, then the demand for risk is increasing in the Sharpe ratio.

Volatility of returns. The volatility of returns is given by

$$\sigma_s[R_p(X,s,s')] = \frac{x_s}{\beta} \frac{\sigma_s[x_{s'}]E'(X,s)^{\frac{\alpha}{1+\nu-\alpha}}}{x_s E^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu} E^{\frac{1+\nu}{1+\nu-\alpha}}}, \tag{96}$$

which is increasing in $E'(X,s)$ and decreasing in x_s.

Note that we can write the coefficient of variation of returns as follows:

$$\frac{\sigma_s[R_p(X,s,s')]}{\mathbb{E}_s[R_p(X,s,s')]} = \frac{\sigma_s[x']}{\mathbb{E}_s[x_{s'}]} \frac{\mathbb{E}_s[x_{s'}]E'(X,s)^{\frac{\alpha}{1+\nu-\alpha}}}{\mathbb{E}_s[x_{s'}]E'(X,s)^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu} E'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}}}, \tag{97}$$

so the presence of labor amplifies the volatility of returns.

Equilibrium. Combining supply and demand for risk, we obtain

$$\frac{x_s}{\beta} \frac{\sigma_s[x_{s'}]E'(X,s)^{\frac{\alpha}{1+\nu-\alpha}}}{x_s E^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu} E^{\frac{1+\nu}{1+\nu-\alpha}}} = \frac{1+\nu-\alpha}{1+\nu} \frac{x_s}{\beta} \frac{\sigma_s[x_{s'}]E'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}}}{x_s E^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu} E^{\frac{1+\nu}{1+\nu-\alpha}}} \left[\frac{p_{sH}(X,s)}{E'(X,s) - x_L} - \frac{p_{sL}(X,s)}{x_H - E'(X,s)} \right]. \tag{98}$$

The left-hand side is strictly increasing in $E'(X,s)$, while the right-hand side is strictly decreasing in $E'(X,s)$ in the interval $x_L < E'(X,s) < x_H$. The right-hand side converges to $+\infty$ as $E'(X',s)$ approaches x_L from above, and it converges to $-\infty$ as $E'(X,s)$ approaches x_H from below. Therefore, there exists a unique value of $E'(X,s)$ solving the equation above in this interval. Note that the two curves intersect again for $E'(X,s) > x_H$, which can be seen by noticing that the right-hand is decreasing in $E'(X,s)$ for $E'(X,s) > x_H$ and converges to $+\infty$ as $E'(X,s)$ approaches x_H from above. Therefore, the economically relevant solution corresponds to the smallest of the two points of intersection.

Rearranging the expression above, we obtain

$$1 = \frac{1+\nu-\alpha}{1+\nu} E'(X,s) \frac{p_{sH}(X,s)(x_H - E'(X,s)) - p_{sL}(X,s)(E'(X,s) - x_L)}{(E'(X,s) - x_L)(x_H - E'(X,s))}. \tag{99}$$

We then obtain a quadratic equation for $E'(X,s)$:

$$\frac{\alpha}{1+\nu}E'(X,s)^2 - \left[\left(1-\frac{1+\nu-\alpha}{1+v}p_{sH}(X,s)\right)x_H + \left(1-\frac{1+\nu-\alpha}{1+v}p_{sL}(X,s)\right)x_L\right]E'(X,s)+x_Lx_H = 0 \tag{100}$$

The equilibrium value is given by the smallest root of the equation above.

□

A.7 Proof of Proposition 4

Proof. We start by deriving the return on the investor's portfolio. Given that markets are complete, there exists a replicating portfolio $\omega^r(X,s)$ such that

$$R_r(X,s,s') = \omega^r(X,s)R_p(X,s,s') + (1-\omega^r(X,s))R_b(X,s) \Rightarrow R_r^e(X,s,s') = \omega^r(X,s)R_p^e(X,s,s'), \tag{101}$$

where $\omega^r(X,s) = \frac{\sigma_s[R_r(X,s,s')]}{\sigma_s[R_p(X,s,s')]} = \frac{\Delta R_r(X,s)}{\Delta R_p(X,s)}$ We can then write the return on the portfolio of investor i as follows:

$$R_{i,n}(X,s,s') = \omega_i(X,s)R_{r,t}^e(X,s,s') + R_b(X,s) = \omega_i(X,s)\frac{\Delta R_r(X,s)}{\Delta R_p(X,s)}R_{p,t}^e(X,s,s') + R_b(X,s). \tag{102}$$

Using condition (17) and the expression for the SDF, we obtain

$$R_{i,n}^e(X,s,s') = R_b(X,s)\left[\frac{p_{sH}^i}{E'(X,s)-x_L} - \frac{p_{sL}^i}{x_H - E'(X,s)}\right]\frac{\Delta x R_p^e(X,s,s')}{\Delta R_p(X,s)} \tag{103}$$

$$= R_b(X,s)\left[\frac{p_{sH}^i}{E'(X,s)-x_L} - \frac{p_{sL}^i}{x_H - E'(X,s)}\right](x_{s'} - E'(X,s)). \tag{104}$$

The return on the portfolio is then given by

$$R_{i,n}(X,s,L) = \frac{\Delta x R_b(X,s)}{x_H - E'(X,s)}p_{sL}^i, \qquad R_{i,n}(X,s,H) = \frac{\Delta x R_b(X,s)}{E'(X,s)-x_L}p_{sH}^i. \tag{105}$$

Wealth share dynamics. The share of wealth of investor i is given by

$$\eta_i'(X,s,s') = \frac{\eta_i R_{i,n}(X,s,s')}{\sum_{j=1}^I \eta_j R_{j,n}(X,s,s')} = \frac{\eta_i p_{ss'}^i}{\sum_{j=1}^I \eta_j p_{ss'}^j} = \eta_i \frac{p_{ss'}^i}{p_{ss'}(X)}. \tag{106}$$

Long-run wealth dynamics. Note that the wealth share is a (bounded) martingale under market beliefs:

$$p_{sH}(X)\eta_i'(X,s,H) + p_{sL}(X)\eta_i'(X,s,L) = \eta_i(p^i_{sH} + p^i_{sL}) = \eta_i. \tag{107}$$

Therefore, from the martingale convergence theorem, the wealth share of investor i converges. This implies that, for every ϵ, there exits T such that

$$|\eta_{i,T+1} - \eta_{i,T}| < \epsilon \iff \eta_i \left| \frac{p^i_{ss'} - p_{ss'}(X)}{p_{ss'}(X)} \right| < \epsilon, \tag{108}$$

almost surely, where the economy is at the state (X, s) at period T.

This implies that either η_i converges to zero or $p^i_{ss'}$ converges to $p_{ss}(X)$. If $p^i_{ss'} \neq p^j_{ss'}$ for any $i, j \in \mathcal{I}$, $i \neq j$, then the wealth share of a single investor converges to one.

By definition, $p^i_{sH} > p_{sH}(X)$ for an optimistic investor in state (X, s), then the wealth share of optimists increase in the good state and decline in the bad state. This implies that market beliefs evolve according to

$$p_{s'H}(X') = \sum_{i=1}^{I} \eta_i' p^i_{s'H} = \sum_{i=1}^{I} \eta_i \frac{p^i_{ss'}}{p_{ss'}(X)} p^i_{s'H}, \tag{109}$$

where

$$p_{HH}(X') = \sum_{i=1}^{I} \eta_i \frac{p^i_{sH}}{p_{sH}(X)} p^i_{HH} \geq p_{sH}(X), \qquad p_{LH}(X') = \sum_{i=1}^{I} \eta_i \frac{p^i_{sL}}{p_{sL}(X)} p^i_{LH} \leq p_{sH}(X). \tag{110}$$

This implies that the relative wealth share for investors i and j is given by

$$\frac{\eta_i'(X,s,s')}{\eta_j'(X,s,s')} = \frac{\eta_i}{\eta_j} \frac{p^i_{ss'}}{p^j_{ss'}}. \tag{111}$$

Suppose investor j beliefs coincides with the objective measure. Then, the ratio above is a martingale:

$$\mathbb{E}_s \left[\frac{\eta_i'}{\eta_j'} \right] = p_{sL} \frac{\eta_i}{\eta_j} \frac{p^i_{sL}}{p_{sL}} + p_{sH} \frac{\eta_i}{\eta_j} \frac{p^i_{sH}}{p_{sH}} = \frac{\eta_i}{\eta_j}. \tag{112}$$

If the wealth of investor j is bounded away from zero, then the above martingale is bounded and, from the martingale convergence theorem, it converges almost surely. □

A.8 Proof of Corollary 2

Proof. Consider an economy that starts at $s = H$ with wealth distribution $\{\eta_i\}_{i=1}^{I}$ which switches to the low state after either one period (early transition) or two periods (late transition). Market beliefs on the low state in the case of an early transition are given by

$$p_{LH}(X') = \sum_{i=1}^{I} \eta_i \frac{p^i_{HL}}{p_{HL}(X)} p^i_{LH}, \tag{113}$$

and market beliefs on the low state in the case of a late transition are given by

$$p_{LH}(X'') = \sum_{i=1}^{I} \eta'_i \frac{p^i_{HL}}{p_{HL}(X')} p^i_{LH}, \tag{114}$$

where $\eta'_i = \eta_i \frac{p^i_{HH}}{p_{HH}(X)}$.

Note that if investor i is optimistic, $p^i_{HH} > p_{HH}(X)$, then $\eta'_i > \eta_i$ and $p^{-i}_{HL}(X') \leq p^{-i}_{HL}(X)$, where $p^{-i}_{HL}(X) \equiv \frac{1}{1-\eta_i} \sum_{j \neq i} \eta_j p^j_{HL}$. This implies that the following inequality holds:

$$\eta'_i \frac{p^i_{HL}}{p_{HL}(X')} = \frac{\eta'_i p^i_{HL}}{\eta'_i p^i_{HL} + (1-\eta'_i) p^{-i}_{HL}(X')} > \frac{\eta_i p^i_{HL}}{\eta_i p^i_{HL} + (1-\eta_i) p^{-i}_{HL}(X)} = \eta_i \frac{p^i_{HL}}{p_{HL}(X)}. \tag{115}$$

Therefore, there is more weight on the beliefs of investors who were optimistic in the original state in the case of a late transition. In the case of rank-preserving beliefs, these agents are also optimistic in the low state, so the market is more optimistic under a late transition:

$$p_{LH}(X'') > p_{LH}(X'). \tag{116}$$

Alternatively, the market is now more pessimistic after a late transition in the case of rank-alternating beliefs:

$$p_{LH}(X'') < p_{LH}(X'). \tag{117}$$

A similar argument shows that, under rank-preserving beliefs, the market is more pessimistic after a late transition when the economy starts at state $s = L$:

$$p_{HH}(X'') = \sum_{i=1}^{I} \eta'_i \frac{p^i_{LH}}{p_{LH}(X')} p^i_{HH} < \sum_{i=1}^{I} \eta_i \frac{p^i_{LH}}{p_{LH}(X)} p^i_{HH} = p_{HH}(X), \tag{118}$$

where $\eta'_i = \eta_i \frac{p^i_{LL}}{p_{LL}(X)}$. Alternatively, the market is more optimistic under a late transition in the case of rank-alternating beliefs.

□

A.9 Proof of Proposition 5

Proof. Volume □

B Derivations

B.1 Markov equilibrium

From the market clearing for bonds and the expression for $B_{i,t}$ from the Lemma **??**, we obtain

$$\sum_{i=1}^{I} \mu_i N_{i,t} = Q_t + \mathcal{H}_t + A_t h_t^{\alpha} - \xi_t \frac{h_t^{1+\nu}}{1+\nu},$$

where $\mathcal{H}_t = \sum_{i=1}^{I} \mu_i \mathcal{H}_{i,t}$ and using the market clearing condition for stocks and goods.

Using the pricing condition for stocks and bonds, we obtain

$$\begin{aligned}\sum_{i=1}^{I} \mu_i N_{i,t} &= \mathbb{E}_t\left[\sum_{k=1}^{\infty} \Lambda_{t,t+k}\pi_{t+k}\right] + \mathbb{E}_t\left[\sum_{k=1}^{\infty} \Lambda_{t,t+k}\left(W_{t+k}h_{t+k} - \xi_{t+k}\frac{h_{t+k}^{1+\nu}}{1+\nu}\right)\right] + A_t h_t^{\alpha} - \xi_t \frac{h_t^{1+\nu}}{1+\nu}, \\ &= \mathbb{E}_t\left[\sum_{k=0}^{\infty} \Lambda_{t,t+k}\left(A_{t+k}h_{t+k}^{\alpha} - \xi_{t+k}\frac{h_{t+k}^{1+\nu}}{1+\nu}\right)\right] = A_{t-1}P_t. \end{aligned} \tag{119}$$

We can write the market clearing for goods as follows:

$$\sum_{i=1}^{I} \frac{\mu_i N_{i,t}}{A_{t-1}P_t}\frac{C_{i,t}}{N_{i,t}} = \frac{A_t h_t^{\alpha} - \xi_t \frac{h_t^{1+\nu}}{1+\nu}}{A_{t-1}P_t}. \tag{120}$$

Using the definition of $\eta_{i,t}$ and the result $\sum_{i=1}^{I} \mu_i N_{i,t} = A_{t-1}P_t$, we obtain the market clearing condition for goods in recursive notation:

$$\sum_{i=1}^{I} \eta_i c_i(X,s) = \frac{x_s h(E)^{\alpha} - \xi \frac{h(E)^{1+\nu}}{1+\nu}}{P(X,s)}. \tag{121}$$

Multiplying the market clearing condition for stocks by Q_t and using the expression for $Q_t S_{i,t}$ given in Lemma **??**, we obtain

$$\sum_{i=1} \mu_i \omega_{i,t}(N_{i,t} - \tilde{C}_{i,t}) = \omega_{e,t} Q_t + \omega_{h,t} \mathcal{H}_t, \tag{122}$$

using the fact that $\mathcal{H}_{i,t} = \mathcal{H}_t$ and defining $\omega_{h,t} \equiv \omega_{h_i,t}$.

We will show next that we can write the right-hand side in the expression above as $\omega_{e,t} Q_t + \omega_{h,t} \mathcal{H}_t = Q_t + \mathcal{H}_t$. To show this fact, we start by the considering the return of holding the aggregate amount of stocks and human wealth:

$$
\begin{aligned}
R_{e,t+1} Q_t + R_{h,t+1} \mathcal{H}_t &= [Q_{t+1} + A_{t+1} h_{t+1}^{\alpha} - W_{t+1} h_{t+1}] + \left[\mathcal{H}_{t+1} + W_{t+1} h_{t+1} - \xi_{t+1} \frac{h_{t+1}^{1+\nu}}{1+\nu} \right] \\
&= A_t P_{t+1} = R_{r,t+1} \left[A_{t-1} P_t - \left(A_t h_t^{\alpha} - \xi_t \frac{h_t^{1+\nu}}{1+\nu} \right) \right] \\
&= R_{r,t+1} \left(Q_t + \mathcal{H}_t \right).
\end{aligned} \tag{123}
$$

This implies the following condition among excess returns:

$$
\begin{aligned}
R_{e,t+1}^e Q_t + R_{h,t+1}^e \mathcal{H}_t &= R_{r,t+1}^e \left(Q_t + \mathcal{H}_t \right) \\
R_{r,t+1}^e \omega_{e,t} Q_t + R_{r,t+1}^e \omega_{h,t} \mathcal{H}_t &= R_{r,t+1}^e \left(Q_t + \mathcal{H}_t \right),
\end{aligned} \tag{124}
$$

which gives the desired result $\omega_{e,t} Q_t + \omega_{h,t} \mathcal{H}_t = Q_t + \mathcal{H}_t$, where we used the fact that $R_{k,t+1}^e = \omega_{k,t} R_{r,t+1}^e$ for $k \in \{e, h\}$.

Using the result above, we can write the market clearing condition on stocks as follows:

$$
\begin{aligned}
\sum_{i=1} \frac{\mu_i N_{i,t} (1 - c_{i,t})}{Q_t + \mathcal{H}_t} \omega_{i,t} &= 1 \\
\sum_{i=1} \frac{\eta_{i,t} A_{t-1} P_t (1 - c_{i,t})}{A_{t-1} P_t - \left(A_t h_t^{\alpha} - \xi_t \frac{h_t^{1+\nu}}{1+\nu} \right)} \omega_{i,t} &= 1.
\end{aligned} \tag{125}
$$

Using the fact that $\frac{A_t h_t^{\alpha} - \xi_t \frac{h_t^{1+\nu}}{1+\nu}}{A_{t-1} P_t} = \sum_{i=1}^{I} \eta_{i,t} c_{i,t}$, we obtain

$$
\sum_{i=1} \tilde{\eta}_{i,t} \omega_{i,t} = 1, \tag{126}
$$

where $\tilde{\eta}_{i,t} \equiv \frac{\eta_i (1 - c_{i,t})}{\sum_{j=1}^{I} \eta_j (1 - c_{j,t})}$.

B.2 Trading volume

Turnover. We start by deriving our measure of trading volume. We focus on the case $\nu = 0$, such that $\mathcal{H}_{i,t} = 0$ and the surplus claim coincides with a claim on the firms' profits.[25] In this case, the portfolio share of stocks for a type-i investor is defined as $\omega_{i,t} \equiv \frac{Q_t S_{i,t}}{N_{i,t}(1-c_{i,t})}$, so $S_{i,t} = \frac{\omega_{i,t} N_{i,t}(1-c_{i,t})}{Q_t}$. Given that $1 - c_{i,t} = \beta$ and $Q_t = \beta P_t$, we obtain $\mu_{i,t} S_{i,t} = \frac{\omega_{i,t}\mu_i N_{i,t}}{P_t} = \omega_{i,t}\eta_{i,t}$. Shares traded by type-$i$ investors are given by $\mu_{i,t}|S_{i,t} - S_{i,t-1}| = |\omega_{i,t}\eta_{i,t} - \omega_{i,t-1}\eta_{i,t-1}|$. Trading volume is then given by

$$\tau_t = \frac{1}{2}\sum_{i=1}^{I} |\omega_{i,t}\eta_{i,t} - \omega_{i,t-1}\eta_{i,t-1}|. \tag{127}$$

In recursive notation, we can write

$$\tau(X, s, s') = \frac{1}{2}\sum_{i=1}^{I} |\omega_i(X', s')\eta_i'(X, s, s') - \omega_i(X, s)\eta_i|, \tag{128}$$

where $X' = \chi(X, s, s')$ and $\eta_i'(X, s, s') = \eta_i \frac{p^i_{ss'}}{p_{ss'}(X)}$.

Solving for the portfolio share. Using the expression for the economy-wide SDF and Equation (17), we can write the portfolio share as follows

$$\omega_i(X, s) = p^i_{sH} \frac{R_b(X, s)}{|R^e_r(X, s, L)|} - p^i_{sL} \frac{R_b(X, s)}{R^e_r(X, s, H)}. \tag{129}$$

Using the fact that $\nu = 0$, the return on the risky and riskless assets can be written as follows:

$$R^e_r(X, s, s') = \frac{x_s}{\beta} \frac{(x_{s'} - E'(X, s))E'(X, s)^{\frac{\alpha}{1-\alpha}}}{x_s E^{\frac{\alpha}{1-\alpha}} - \alpha E^{\frac{1}{1-\alpha}}}, \qquad R_b(X, s) = (1-\alpha)\frac{x_s}{\beta} \frac{E'(X, s)^{\frac{1}{1-\alpha}}}{x_s E^{\frac{\alpha}{1-\alpha}} - \alpha E^{\frac{1}{1-\alpha}}}, \tag{130}$$

Combining the previous expressions, we obtain

$$\omega_i(X, s) = (1-\alpha)\left[p^i_{sH}\frac{E'(X, s)}{E'(X, s) - x_L} - p^i_{sL}\frac{E'(X, s)}{x_H - E'(X, s)}\right], \tag{131}$$

which is strictly decreasing in $E'(X, s)$ and $\omega_i(X, s) > 1$ if and only if $p^i_{sH} > p_{sH}(X)$.

Turnover is then given by

$$\tau(X, s, s') = (1-\alpha)\sum_{i=1}^{I}\eta_i \left| \left(\frac{p^i_{s'H}E'(X', s')}{E'(X', s') - x_L} - \frac{p^i_{s'L}E'(X', s')}{x_H - E'(X', s')}\right)\frac{p^i_{ss'}}{p_{ss'}(X)} - \left(\frac{p^i_{sH}E'(X, s)}{E'(X, s) - x_L} - \frac{p^i_{sL}E'(X, s)}{x_H - E'(X, s)}\right)\right| \tag{132}$$

[25]Notice that $W_t = \xi_t$ when $\nu = 0$, so $\xi_t \frac{h_t^{1+\nu}}{1+\nu} = W_t h_t$. This implies that P_t corresponds to the present discounted value of firms' profits and that human wealth is equal to zero for all investors.

Perturbation. It is useful to parameterize the dispersion in beliefs as follows:

$$p^i_{ss'} = p^*_{ss'} + \epsilon\delta^i_{ss'}, \tag{133}$$

where $\delta^i_{sH} + \delta^i_{sL} = 0$. If $\epsilon = 0$, then there is no belief heterogeneity and $\tau(X, s, s') = 0$. We consider next how turnover depends on belief heterogeneity for small deviations of this benchmark, that is, for ϵ close to zero.

Notice that all equilibrium variables now depend on ϵ. For instance, the average probability of the high state can be written as

$$p_{sH}(X; \epsilon) = p^*_{sH} + \delta_{sH}(X)\epsilon + \mathcal{O}(\epsilon^2), \tag{134}$$

where $\delta_{sH}(X) \equiv \sum_{i=1} \eta_i \delta^i_{sH}$. Risk-neutral expectation of productivity growth is a function of $E'(X, s; \epsilon) = f_s(p_{sH}(X))$, where $f_s(p)$ satisfies the condition

$$1 = (1-\alpha)\left[p\frac{f_s(p)}{f_s(p) - x_L} - (1-p)\frac{f_s(p)}{x_H - f_s(p)}\right] \Rightarrow f'_s(p) = \frac{\frac{f_s(p)}{f_s(p)-x_L} + \frac{f_s(p)}{x_H - f_s(p)}}{p\frac{x_L}{(f(p)-x_L)^2} + (1-p)\frac{x_H}{(x_H - f_s(p))^2}}. \tag{135}$$

Let $E^*(X, s) \equiv E'(X, s; 0)$ denote the value of $E'(X, s)$ when $\epsilon = 0$. In this case, we can drop the dependence on X and simply write $E^*(s)$, as $E'(X, s)$ would only depend on the state s. We can then expand $E'(X, s; \epsilon)$ in ϵ to obtain:

$$E'(X, s; \epsilon) = E^*(s) + \tilde{E}(X, s)\epsilon + \mathcal{O}(\epsilon^2), \tag{136}$$

where $\tilde{E}(X, s) = f'(p^*_{sH}) \sum_{i=1}^{I} \eta_i \delta^i_{sH}$, where $f'(\cdot) > 0$.

We can then write the portfolio share of investor i as follows

$$\omega_i(X, s; \epsilon) = 1 + \left[\theta_{\omega,1}(s)\delta^i_{sH} - \theta_{\omega,2}(s)\delta_{sH}(X)\right]\epsilon + \mathcal{O}(\epsilon^2), \tag{137}$$

where $\theta_{\omega,1}(s) > 0$ and $\theta_{\omega,2}(s) > 0$

$$\theta_{\omega,1}(s) \equiv (1-\alpha)\left(\frac{E^*(s)}{E^*(s) - x_L} + \frac{E^*(s)}{x_H - E^*(s)}\right) \tag{138}$$

$$\theta_{\omega,2}(s) \equiv (1-\alpha)\left[\frac{p^*_{sH} x_L}{(E^*(s) - x_L)^2} + \frac{p^*_{sL} x_H}{(x_H - E^*(s))^2}\right] f'(p^*_{sH}). \tag{139}$$

Using the expression for $f'(\cdot)$, we obtain that $\theta_{\omega,1} = \theta_{\omega,2}$. We can then write $\omega_i(X, s; \epsilon)$

as follows:

$$\omega_i(X,s;\epsilon) = 1 + \theta_{\omega,1}(s)\left[\delta^i_{sH} - \delta_{sH}(X)\right]\epsilon + \mathcal{O}(\epsilon^2), \tag{140}$$

The evolution of wealth is given by

$$\eta_i'(X,s,s';\epsilon) = \eta_i + \eta_i \frac{\delta^i_{ss'} - \delta_{ss'}(X)}{p^*_{ss'}}\epsilon + \mathcal{O}(\epsilon^2) \tag{141}$$

Let $p_H(X,s,s';\epsilon) = \sum_{i=1}^I \eta_i'(x,s,s';\epsilon)p^i_{s'H}$ denote the market-implied probability of the high state after a transition to state s', then

$$p_H(X,s,s';\epsilon) = p^*_{s'H} + \delta_{s'H}(X)\epsilon + \mathcal{O}(\epsilon^2), \tag{142}$$

where $\delta_{s'H}(X) \equiv \sum_{i=1}^I \eta_i \frac{\delta^i_{ss'} - \delta_{ss'}(X)}{p^*_{ss'}} p^*_{s'H} + \sum_{i=1}^I \eta_i \delta^i_{s'H} = \sum_{i=1}^I \eta_i \delta^i_{s'H}$.

The portfolio share next period is given by

$$\omega_i'(X,s,s';\epsilon) = 1 + \theta_{\omega,1}(s')\left[\delta^i_{s'H} - \delta_{s'H}(X)\right]\epsilon + \mathcal{O}(\epsilon^2). \tag{143}$$

Investor i's net purchases of shares is given by

$$\Delta S_i(X,s,s';\epsilon) = \eta_i \left[\frac{\delta^i_{ss'} - \delta_{ss'}(X)}{p^*_{ss'}} + \theta_{\omega,1}(s')\left(\delta^i_{s'H} - \delta_{s'H}(X)\right)\right]\epsilon - \theta_{\omega,1}(s)\eta_i\left[\delta^i_{sH} - \delta_{sH}(X)\right]\epsilon + \mathcal{O}(\epsilon^2) \tag{144}$$

For simplicity, suppose that investors believe productivity growth to be iid in the reference economy, that is, $p^*_{Ls'} = p^*_{Hs'}$. We can then write the

$$\Delta S_i(X,s,s';\epsilon) = \left[\underbrace{\Delta\tilde{\omega}_i(X,s,s')\eta_i}_{\text{change-in-beliefs effect}} + \underbrace{\Delta\tilde{\eta}_i(X,s,s')}_{\text{rebalancing effect}}\right]\epsilon + \mathcal{O}(\epsilon^2), \tag{145}$$

where

$$\Delta\tilde{\omega}_i(X,s,s') \equiv \theta_{\omega,1}\left[\left(\delta^i_{s'H} - \delta_{s'H}(X,s)\right) - \left(\delta^i_{sH} - \delta_{sH}(X)\right)\right] \tag{146}$$

$$\Delta\tilde{\eta}_i(X,s,s') \equiv \eta_i \frac{\delta^i_{ss'} - \delta_{ss'}(X)}{p^*_{ss'}}. \tag{147}$$

The change-in-beliefs effect captures the fact that investors buy (sell) the asset if they are more optimistic (pessimistic) in state s' than in state s. This effect is equal to zero when $s = s'$. The rebalancing effect captures the fact that investors buy (sell) the asset if their share of wealth increases (decreases) after the economy switches to state s', even if there is no change in the desired portfolio share ω_i.

Turnover is given by

$$\tau(X, s, s'; \epsilon) = \frac{1}{2} \sum_{i=1}^{I} \eta_i \left| \frac{\tilde{\delta}^i_{ss'}}{p^*_{ss'}} + \kappa_\omega \left(\tilde{\delta}^i_{s'H} - \tilde{\delta}^i_{sH} \right) \right| \epsilon + \mathcal{O}(\epsilon^2), \tag{148}$$

where $\tilde{\delta}^i_{ss'} = \delta^i_{ss'} - \delta_{ss'}(X)$.

Suppose $s = s'$, then

$$\tau(X, s, s'; \epsilon) = \frac{1}{2} \sum_{i=1}^{I} \eta_i \frac{|\tilde{\delta}^i_{ss'}(X)|}{p^*_{ss'}} \epsilon + \mathcal{O}(\epsilon^2) \tag{149}$$

$$= \frac{1}{2} \left[\sum_{i=1}^{I} \eta_i \frac{\tilde{\delta}^i_{ss'}(X)}{p^*_{ss'}} \mathbf{1}_{\tilde{\delta}^i_{ss'}(X) \geq 0} - \sum_{i=1}^{I} \eta_i \frac{\tilde{\delta}^i_{ss'}(X)}{p^*_{ss'}} \mathbf{1}_{\tilde{\delta}^i_{ss'} < 0} \right] \epsilon + \mathcal{O}(\epsilon^2) \tag{150}$$

$$= \frac{1}{2} \left[\eta_B \frac{\tilde{\delta}^B_{ss'}(X)}{p^*_{ss'}} + \eta_S \frac{|\tilde{\delta}^S_{ss'}(X)|}{p^*_{ss'}} \right] \epsilon + \mathcal{O}(\epsilon^2). \tag{151}$$

where

$$\eta_B \equiv \sum_{i=1}^{I} \eta_i \mathbf{1}_{\tilde{\delta}^i_{ss'}(X) \geq 0}, \qquad \tilde{\delta}^B_{ss'}(X) \equiv \frac{1}{\eta_B} \sum_{i=1}^{I} \eta_i \tilde{\delta}^i_{ss'}(X) \mathbf{1}_{\tilde{\delta}^i_{ss'}(X) \geq 0}, \tag{152}$$

$$\eta_S \equiv \sum_{i=1}^{I} \eta_i \mathbf{1}_{\tilde{\delta}^i_{ss'}(X) < 0}, \qquad \tilde{\delta}^S_{ss'}(X) \equiv \frac{1}{\eta_S} \sum_{i=1}^{I} \eta_i \tilde{\delta}^i_{ss'}(X) \mathbf{1}_{\tilde{\delta}^i_{ss'}(X) < 0}. \tag{153}$$

We can write turnover in this case as follows

$$\tau(X, s, s'; \epsilon) = \eta_B \eta_S \frac{\delta^B_{ss'}(X) + |\delta^S_{ss'}(X)|}{p^*_{ss'}} \epsilon + \mathcal{O}(\epsilon^2), \tag{154}$$

using the fact that $\delta_{ss'}(X) = \eta_B \delta^B_{ss'}(X) + \eta_S \delta^S_{ss'}(X)$.

Heterogeneous persistence. We consider next the special case where investors agree about the unconditional mean of x, but they disagree about the persistence of the aggregate productivity growth.

The stationary distribution of beliefs for investor i is given by

$$p^i_L = \frac{p^i_{HL}}{p^i_{LH} + p^i_{HL}}. \tag{155}$$

We assume that p^i_L is equalized across investors, so all investors agree about the unconditional mean of x_t. Note this implies that the likelihood ratio p^i_{LH}/p^i_{HL} is equalized across investors. The unconditional mean is given by

$$\overline{x} = \frac{p^i_{HL}}{p^i_{LH} + p^i_{HL}} x_L + \frac{p^i_{LH}}{p^i_{LH} + p^i_{HL}} x_H. \tag{156}$$

The expected value of x_{t+1} relative to the mean $\overline{x}$ conditional on $x_t = x_L$ is given by

$$\mathbb{E}_i[x_{t+1} - \overline{x} | x_t = x_L] = p^i_{LL}(x_L - \overline{x}) + p^i_{LH}(x_H - \overline{x}) \tag{157}$$

$$= \left[1 + p^i_{LH} \frac{x_H - x_L}{x_L - \overline{x}}\right](x_L - \overline{x}) \tag{158}$$

$$= \left[1 - (p^i_{LH} + p^i_{HL})\right](x_L - \overline{x}), \tag{159}$$

using the fact that $\overline{x} - x_L = \frac{p^i_{LH}}{p^i_{LH} + p^i_{HL}}(x_H - x_L)$

We obtain a similar expression conditioning on $x_t = x_H$ instead:

$$\mathbb{E}_i[x_{t+1} - \overline{x} | x_t = x_H] = p^i_{HL}(x_L - \overline{x}) + p^i_{HH}(x_H - \overline{x}) \tag{160}$$

$$= \left[1 - p^i_{HL} \frac{x_H - x_L}{x_H - \overline{x}}\right](x_H - \overline{x}) \tag{161}$$

$$= \left[1 - (p^i_{LH} + p^i_{HL})\right](x_H - \overline{x}), \tag{162}$$

using the fact that $x_H - \overline{x} = \frac{p^i_{HL}}{p^i_{LH} + p^i_{HL}}(x_H - x_L)$.

Let $\hat{x}_t = x_t - \overline{x}$, we can then write

$$\mathbb{E}_i[\hat{x}_{t+1} | \hat{x}_t] = \theta_i \hat{x}_t, \tag{163}$$

where $\theta_i \equiv 1 - (p^i_{LH} + p^i_{HL}) = p^i_{HH} - p^i_{LH}$.

Given that investors agree about the unconditional mean of x, we are able to pin down beliefs as a function of θ_i:

$$p^i_{LH} = \overline{p}_H(1 - \theta_i), \qquad p^i_{HH} = \overline{p}_H + \overline{p}_L \theta_i. \tag{164}$$

Corollary. Under the assumption investors agree about the unconditional mean of x_t, we have that

$$p^i_{LH} - p_{LH}(X) = -\overline{p}_H(\theta_i - \theta(X)), \qquad p^i_{HH} - p_{HH}(X) = \overline{p}_L(\theta_i - \theta(X)), \tag{165}$$

where $\overline{\theta}(X) \equiv \sum_{i=1}^{I} \eta_i \theta_i$.

Notice that we have that $\tilde{\delta}^i_{ss'}(X)\epsilon = p^i_{ss'} - p_{ss'}(X)$, which gives us

$$\tilde{\delta}^i_{LH}(X)\epsilon = -\overline{p}_H(\theta_i - \theta(X)), \qquad \tilde{\delta}^i_{HH}(X)\epsilon = (1 - \overline{p}_H)(\theta_i - \theta(X)). \tag{166}$$

We can then write turnover in the case $s = L$ and $s' = H$ as follows:

$$\tau(X, L, H; \epsilon) = \frac{1}{2} \left| \kappa_\omega - \frac{\overline{p}_H}{p^*_H} \right| \sum_{i=1}^{I} \eta_i |\theta_i - \theta(X)| + \mathcal{O}(\epsilon^2). \tag{167}$$

Consider now the case $s = H$ and $s' = L$:

$$\tau(X, H, L; \epsilon) = \frac{1}{2} \left| \kappa_\omega + \frac{\overline{p}_L}{p^*_L} \right| \sum_{i=1}^{I} \eta_i |\theta_i - \theta(X)| + \mathcal{O}(\epsilon^2), \tag{168}$$

Suppose now that $s = s' = L$, then

$$\tau(X, H, H; \epsilon) = \frac{1}{2} \left| \frac{\overline{p}_L}{p^*_H} \right| \sum_{i=1}^{I} \eta_i |\theta_i - \theta(X)| \epsilon + \mathcal{O}(\epsilon^2) \tag{169}$$

$$\tau(X, L, L; \epsilon) = \frac{1}{2} \left| \frac{\overline{p}_H}{p^*_L} \right| \sum_{i=1}^{I} \eta_i |\theta_i - \theta(X)| \epsilon + \mathcal{O}(\epsilon^2). \tag{170}$$

C Estimating the Heterogeneity in Beliefs

C.1 The process for realized and expected earnings

Let $i \in \mathcal{I}$ denote a firm-analyst pair. We index both firm-level outcomes and the expectations of the analyst covering this firm by i. We denote (realized) earnings for firm i at period t by $e_{i,t}$ and the first-difference of realized earnings by $\Delta e_{i,t} = e_{i,t} - e_{i,t-1}$.[26] We denote aggregate earnings by e_t and the first-difference of aggregate earnings by Δe_t. Realized earnings follows the process:

$$\Delta e_{i,t} = \beta_i \Delta e_t + u_{i,t}, \tag{171}$$

where $u_{i,t} = \rho_i u_{i,t-1} + \epsilon_{i,t}$ and $\epsilon_{i,t} \sim \mathcal{N}(0, \sigma^2_\epsilon)$. The error term $\epsilon_{i,t}$ is assumed to be i.i.d. and independent of Δe_t. We assume that $\Delta e_{i,t}$ and Δe_t have already been de-meaned, so

[26] As $e_{i,t}$ can potentially be negative, we work with first differences instead of proportional differences, $\frac{\Delta e_{i,t}}{e_{i,t}}$, or log-differences, $\Delta \log(e_{i,t})$. By focusing on first differences, we do not have to drop firms which experience negative earnings, which is a significant fraction of our sample.

we can omit the intercept. We also assume that $\Delta e_{i,t}$ and Δe_t have been normalized to have unit variance.

Given the formulation above, individual earnings depend on aggregate shocks, i.e. shocks that affect aggregate earnings, as well as idiosyncratic shocks, as captured by $u_{i,t}$. The parameters ρ_i controls the persistence of idiosyncratic shocks. Hence, firms are allowed to be heterogeneous on their exposure to the aggregate shock as well as the persistence of idiosyncratic shocks.

We assume that analysts understand that individual earnings follows the process (171), but they potentially disagree on the process followed by aggregated earnings. In particular, we assume that analyst i believe (in a dogmatic fashion) that Δe_t follows the following process:

$$\Delta e_t = \theta_i \Delta e_{t-1} + \nu_{i,t}, \tag{172}$$

where $\nu_{i,t}$ is an i.i.d. process given by $\nu_{i,t} \sim \mathcal{N}(0, \sigma_\nu^2)$. We assume that analysts agree on the unconditional mean for Δe_t, which we normalize to zero. This allow us to focus only on disagreement about the persistence of shocks to aggregate earnings.

The expected change in aggregate earnings using the subjective beliefs of analyst i is given by

$$\mathbb{E}_{i,t}[\Delta e_{t+1}] = \theta_i \Delta e_t, \tag{173}$$

where $\mathbb{E}_{i,t}[\cdot]$ denote the conditional expectation at t according to the subjective beliefs of analyst i.

We assume that Δe_t is perfectly observed by investors at time t, so differences in beliefs are controlled by θ_i. A relatively high value for θ_i implies that analyst i is more optimistic about aggregate earnings after a positive shock and more pessimistic after a negative shock, capturing a form of belief extrapolation.

Notice that expectations of changes in *individual* earnings depend on the degree of persistence of shocks to *aggregate* earnings θ_i:

$$\mathbb{E}_{i,t}[\Delta e_{i,t+1}] = \beta_i \theta_i \Delta e_t + \rho_i u_{i,t}. \tag{174}$$

Equation (174) shows that we can infer properties of the process for subjective beliefs on *aggregate* earnings using information on subjective beliefs about *individual* earnings. This is important as beliefs on aggregate earnings are not directly available.

C.2 Estimation procedure

We show next how to estimate $(\beta_i, \rho_i, \theta_i)$ in two stages. First, we estimate the parameters in Equation (171). In a second stage, we obtain the distribution of θ_i, using Equation (174) and the parameters estimated in the first stage.

First stage. Consider first Equation (171). We can rewrite the process for $\Delta e_{i,t}$ as follows:

$$\Delta e_{i,t} = \beta_i \Delta e_t + \rho_i \left(\Delta e_{i,t-1} - \beta_i \Delta e_{t-1}\right) + \epsilon_{i,t}, \tag{175}$$

where we used the fact that $u_{i,t} = \Delta e_{i,t} - \beta_i \Delta e_t$.

To ensure that $-1 < \rho_i < 1$, we consider the following change of variables. Assume that ρ_i is given by the a non-linear transformation of the parameter $\tilde{\rho}_i \in \mathbb{R}$: $\rho_i = -1 + 2\frac{\exp(\tilde{\rho}_i)}{1+\exp(\tilde{\rho}_i)} \in (-1, 1)$. The parameters $(\beta_i, \tilde{\rho}_i)$ can in principle be estimated using, for instance, non-linear least squares for each company i. We proceed instead by estimating the parameters simultaneously for all i using Bayesian methods. The Bayesian approach is useful as it allow us to regularize the individual estimates and avoid overfitting, which can be a concern in settings where the length of the time series is not particularly long.[27]

Formally, we consider the following multi-level priors:

$$\beta_i \sim \mathcal{N}(\overline{\beta}, \sigma_\beta^2), \qquad \tilde{\rho}_i \sim \mathcal{N}(\overline{\rho}, \sigma_\rho^2), \tag{176}$$

The coefficients $(\overline{\beta}, \overline{\rho})$ and $(\sigma_\beta, \sigma_\rho)$ are referred to as *hyperparameters* and they have their own priors, which are given by

$$\overline{\beta} \sim \mathcal{N}(0, 1.50^2), \qquad \overline{\rho} \sim \mathcal{N}(0, 0.50^2), \tag{177}$$

and the standard-deviation for each parameter is assumed to follow a Half Student-t distribution with 3 degrees of freedom, a standard value for this class of models. These priors are set to be wide enough to capture the range of plausible values for the parameters.

The multi-level structure allow us to obtain a form of adaptive regularization. If (say) σ_β is very large, then the prior on β_i is not very informative, and this would be analogous to estimate β_i independently for each i. If $\sigma_\beta \approx 0$, then we have effectively a pooling estimator, where β_i will be the same for all i. For intermediate values of σ_β, the parameters

[27] This procedure is analogous to a ridge regression, where the estimates are regularized using a L2 penalty (see e.g. Hastie et al., 2009). For a discussion of how regularized regressions can be reinterpreted as a Bayesian procedure, see e.g. Nagel (2021).

are allowed to vary across units, but they are partially shrunk towards the population mean. The shrinkage of the parameters limits the effect of noise or measurement error, as the model is essentially skeptical of extreme values. Because σ_β is also an estimated parameter, the extent to which estimates are regularized is directly informed by the data.[28]

Second stage. Consider next Equation (174), which relates subjective beliefs about individual earnings to realized aggregate and individual earnings. To capture the fact that (subjective) expectations are potentially measured with error, we assume that only a noisy version of the analyst's expectation is observed, which is given by $\hat{\mathbb{E}}_{i,t}[\Delta e_{i,t+1}] = \mathbb{E}_{i,t}[\Delta e_{i,t+1}] + \tilde{w}_{i,t}$. The measurement error $\tilde{w}_{i,t}$ is assumed to be a mean-zero normally distributed i.i.d. process with variance given by σ_w^2. Combining this measurement equation with Equation (174) and isolating the terms estimated in the first stage, we obtain the following estimating equation:

$$z_{i,t} = \alpha_i + \theta_i x_{i,t} + w_{i,t}, \tag{178}$$

where $z_{i,t} \equiv \hat{\mathbb{E}}_{i,t}[\Delta e_{i,t+1}] - \rho_i u_{i,t}$ and $x_{i,t} \equiv \beta_i \Delta e_t$. Notice that $z_{i,t}$ and $x_{i,t}$ are known at this stage, so it only remains to estimate $\theta_{i,t}$.

As before, we use a Bayesian multi-level model to adaptively regularize our estimates. We also consider the transformation $\theta_i = -1 + 2\frac{\exp(\tilde{\theta}_i)}{1+\exp(\tilde{\theta}_i)}$, where $\tilde{\theta}_i \in \mathbb{R}$, such that we can ensure that $\theta_i \in (-1, 1)$. We assume the following prior for $\tilde{\theta}_{i,t}$:

$$\theta_{i,t} \sim \mathcal{N}(\overline{\theta}, \sigma_\theta^2), \tag{179}$$

where $\overline{\theta} \sim \mathcal{N}(0, 0.5^2)$ and σ_θ follows a half Student-t distribution with 3 degrees of freedom.

C.3 Data and estimation results

Data. We use data from I/B/E/S on analysts expectations about firms' future earnings. For firms with coverage of more than one analyst, we use the consensus expectation for that firm. We drop firms with missing values for realized or expected earnings in more than 20% of the sample. We ended up with 579 firms covering the time period from March 1977 until December 2020, with a total of 44,267 company-quarter pairs.

[28]For more details on how multi-level models provide a form of adaptive regularization, see e.g. the discussion in McElreath (2020).

Table 4: Cross-sectional mean and dispersion of parameters

	Estimate	Est.Error	l-95% CI	u-95% CI	Rhat
$\overline{\mathbb{E}}[\beta_i]$	0.03	0.01	0.01	0.04	1.00
$\overline{\mathbb{E}}[\rho_i]$	0.45	0.02	0.41	0.50	1.00
$\overline{\mathbb{E}}[\theta_i]$	-0.48	0.12	-0.72	-0.24	1.00
$\overline{\sigma}[\beta_i]$	0.09	0.01	0.08	0.10	1.00
$\overline{\sigma}[\rho_i]$	0.47	0.02	0.43	0.51	1.00
$\overline{\sigma}[\theta_i]$	0.19	0.13	0.01	0.49	1.00

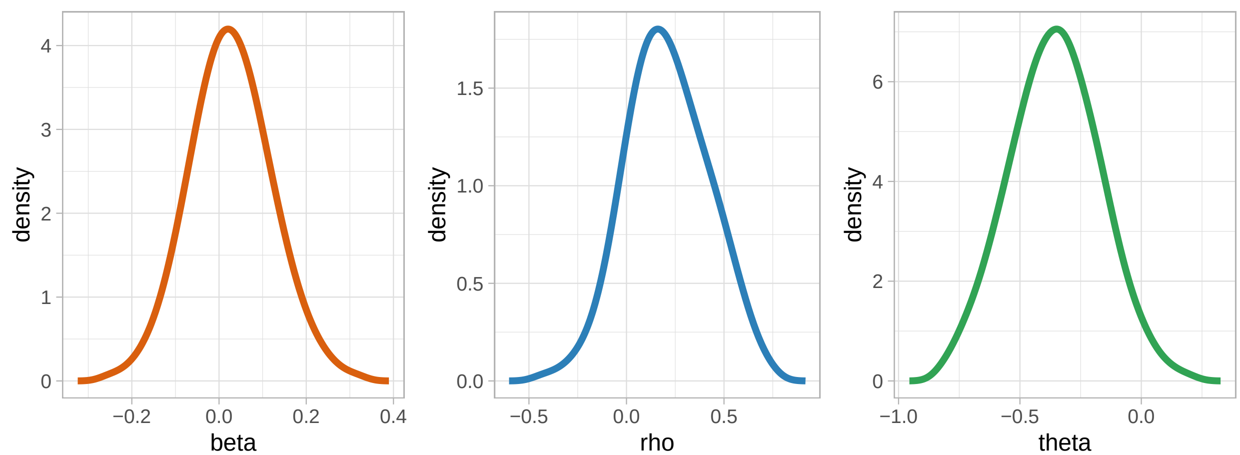
Note: Posterior mean and credible intervals (CI) for the cross-sectional mean, $\overline{\mathbb{E}}[x_i]$, and cross-sectional standard-deviation, $\overline{\sigma}[x_i]$, for parameters $x \in \{\beta, \rho, \theta\}$. Rhat is an indicator of the convergence of the chains during sampling. Rhat = 1 indicates convergence.

Model fitting and results. We sample the model using an extension of Hamiltonian Monte Carlo, the no-U-turn sampler (NUTS) by Hoffman et al. (2014), as implemented in R Stan. Table 4 reports the posterior mean and 95% credible intervals for the cross-sectional mean and dispersion of parameters $(\beta_i, \rho_i, \theta_i)$. Because we have standardized all the variables, the parameter β_i captures the correlation between individual and aggregate earnings. The correlation is close to zero reflecting the fact that typically most of the variation in a company's earnings reflect idiosyncratic shocks. However, there is substantial heterogeneity in this parameter, with the cross-sectional dispersion being three times the average β_i. This can be seen in the left panel of Figure 6, which shows the posterior mean of the kernel density for β_i, where β_i ranges from -0.3 to 0.4. The average autocorrelation coefficient ρ_i is positive, but it is also very dispersed across firms, as shown in the middle panel of Figure 6. Finally, we have that θ_i is on average negative, which is consistent with the fact that Δe_t has a negative autocorrelation. However, the average subjective coefficient of autocorrelation is more negative than its objective counterpart, as $\mathbb{E}[\theta_i] = -0.48$ and we obtain a coefficient of autocorrelation of -0.28 for Δe_t using aggregate data. As before, we observe substantial heterogeneity in θ_i, as shown in the right panel of Figure 6.

C.4 Belief disagreement and stock market turnover

We consider next a measure of belief disagreement. Notice that the expectation of analyst of aggregate earnings growth is given by $\mathbb{E}_i[\Delta e_{t+1}] = \theta_i \Delta e_t$. This motivates our definition of a *disagreement index* DI_t, which corresponds to the cross-sectional dispersion in beliefs

Figure 6: Kernel estimate of cross-sectional distribution of the different parameters



Note: Posterior mean of the kernel density for the cross-section of θ_i (left panel), ρ_i (middle panel), and θ_i (right panel).

about aggregate earnings growth:

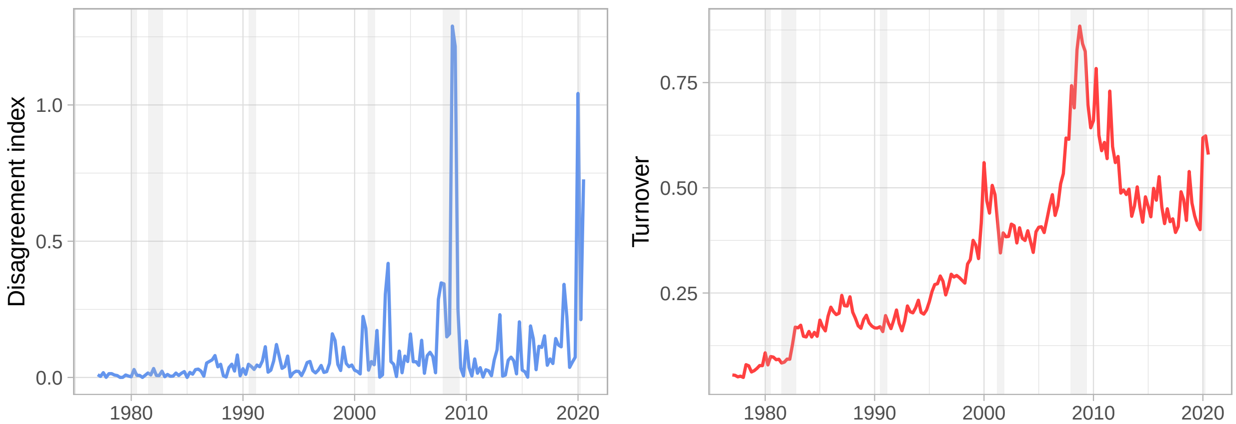
$$DI_t = \underbrace{\overline{\sigma}[\theta_i] \times |\Delta e_t|}_{\overline{\sigma}[\mathbb{E}_i[\Delta e_{t+1}]]}. \tag{180}$$

The disagreement index has two components. First, the cross-sectional dispersion in the parameter θ_i. If all analysts agree on the persistence of aggregate earnings growth, such that $\overline{\sigma}[\theta_i] = 0$, then the disagreement index would be equal to zero. Second, the absolute value of aggregate earnings growth, $|\Delta e_t|$. Given that Δe_t has been already demeaned, $|\Delta e_t|$ captures the distance of aggregate earnings growth to its mean. If aggregate earnings growth is already at its average value, $|\Delta e_t| = 0$, then disagreement on how Δe_t reverts to its plays no role in determining expectations. Therefore, the level of disagreement in the economy depends on the interaction between dispersion in beliefs and deviations of aggregate earnings growth from its mean.

The left panel of Figure 7 shows the time series of the disagreement index. The disagreement index is typically low during normal times and it significantly spikes in periods of crises, where aggregate earnings growth deviates substantially from its average value.

Turnover. One important implication of theories with heterogeneous beliefs is that the level of disagreement is related to the amount of trading in the economy. To test this

Figure 7: Time series of the disagreement index and stock market turnover



Note: Left panel shows the time series of the disagreement index and the right panel shows the time-series of stock market turnover. The smooth line in the right panel is the HP-filter trend of turnover. The vertical bars represent NBER recessions.

implication, we consider next a measure of trading activity, the (value-weighted) stock market turnover.[29] We measure the stock turnover - shares traded divided by shares outstanding - for individual securities on the New York and American Stock Exchanges from January 1977 to December 2021. We measure turnover at the quarterly frequency and compute an aggregate turnover measure using a value-weighted average (similar results are obtained by using an equal-weight measure). The right panel of Figure 7 shows the time series of turnover. We can observe that the turnover level changed significantly over time and that turnover has an important cyclical component.

Belief disagreement and turnover. We consider next the relationship between belief disagreement and turnover. Table 5 shows the result of a time-series regression of turnover on the disagreement index. As shown in Figure 7, the disagreement index series has a few outliers, in particular, during crisis periods. To ensure that the relationship between turnover and disagreement is not driven only by these extreme periods, we consider a sample where we exclude observations where the disagreement index is below the 2.5% percentile or above the 97.5% percentile. Column (1) shows that there is a strong statistically significant association between DI and turnover, where we compute Newey-West standard-errors with four lags. If the disagreement index goes from its 25% percentile to its 75% percentile, turnover increases by 8.0 percentage points, an

[29]For a discussion of turnover as a measure of trading volume and its connection with standard portfolio theory, see Lo and Wang (2010).

Table 5: Regression of turnover on disagreement index

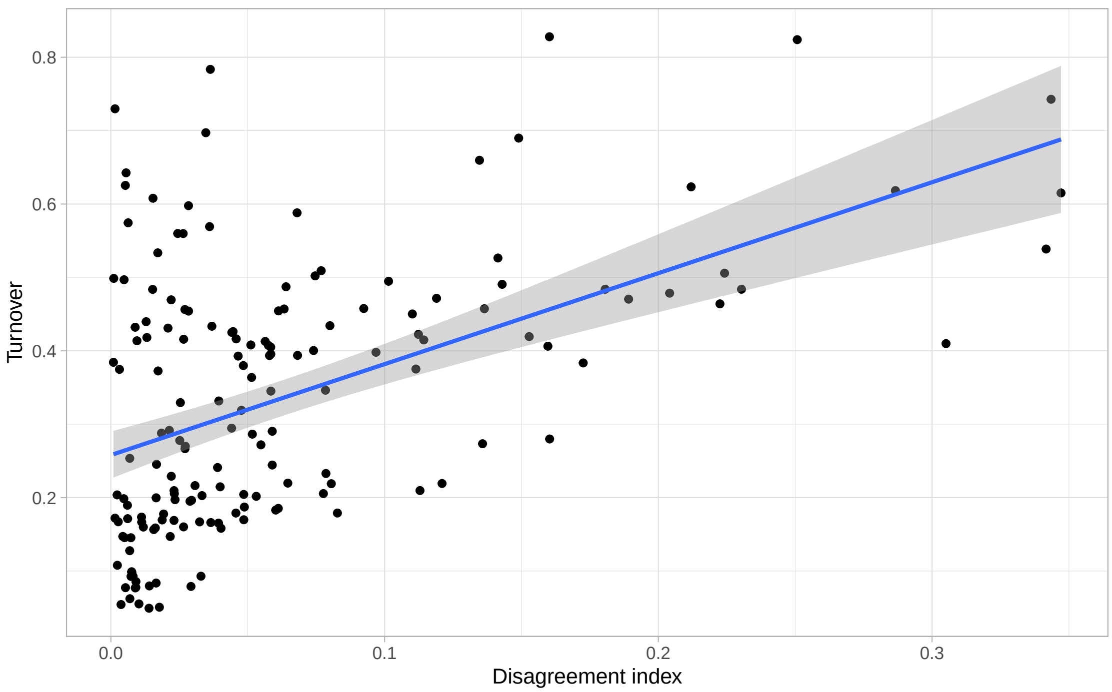
Dependent Variable:		*turnover*	
Model:	(1)	(2)	(3)
Variables			
(Intercept)	0.2580***	0.2420***	0.2549***
	(0.03373)	(0.04375)	(0.0369)
DI	1.239***	1.798**	1.260***
	(0.2277)	(0.6277)	(0.2898)
DI^2		-2.068	-0.6879**
		(1.6920)	(0.2094)
Fit statistics			
Observations	165	165	175
R^2	0.24084	0.24786	0.30386
Adjusted R^2	0.23618	0.23857	0.29576

Newey-West standard-errors in parentheses (4 lags)
*Signif. Codes: ***: 0.01, **: 0.05, *: 0.1*

Note: Columns (1) and (2) .

increase of almost 30%. Column (2) tests whether this relationship is nonlinear by introducing a quadratic term, again in the example where we exclude outliers. We find that the quadratic term is not significant, consistent with a linear relationship. This can be verified visually in Figure 8, which shows the scatterplot of turnover and the disagreement index for this sample. Column (3) shows the regression of turnover on DI and DI^2 for the full sample. We find that the quadratic term is now statistically significant, indicating the necessity of considering a nonlinear relationship to capture the effect of the extreme crisis-level disagreement. The magnitude of the marginal effect of changing DI is similar to the linear case for large of values for the disagreement index. Therefore, we conclude that belief disagreement is strongly associated with stock market turnover.

Figure 8: Scatterplot of the disagreement index and stock market turnover



Note: Scatterplot of disagreement index and turnover for a sample without outliers.

Online Appendix

O1 The case with an arbitrary number of states

O1.1 Environment

We consider an extension of the model in Section 2 where aggregate productivity growth x_t takes N possible values, i.e. $x_t \in \{x^1, x^2, \ldots, x^N\} \equiv \mathcal{X}$, where $x^1 < x^2 < \ldots < x^N$. The objective probability of switching from state $s \in \{1, 2, \ldots, N\} \equiv \mathcal{S}$ to state $s' \in \mathcal{S}$ is denoted by $p_{ss'}$ and the corresponding subjective probability for household $i \in \mathcal{I}$ is denoted by $p^i_{ss'}$. Households can trade Arrow securities that pay off conditional on every possible state. We also assume that households die with probability κ and leave their financial wealth to their child, which will have type j with probability μ_j. This assumption ensures that a non-degenerate stationary distribution of wealth exists. The next proposition provides a characterization of the equilibrium in this N-state economy. The main conclusion is that the results of Section 2 are essentially unchanged in this more general setting.

Proposition O.1 (N-state economy). *Suppose that $x_t \in \mathcal{X}$, where x_t takes N possible values.*

i. The (scaled) household's problem can be written as follows

$$\frac{v_i(X,s)^{1-\psi^{-1}} - 1}{1-\psi^{-1}} = \max_{c_i, R'_{i,n}} (1-\beta) \frac{c_i^{1-\psi^{-1}} - 1}{1-\psi^{-1}} + \beta \frac{\mathbb{E}_i \left[(v_i(X', s') n')^{1-\gamma} \right]^{\frac{1-\psi^{-1}}{1-\gamma}} - 1}{1-\psi^{-1}}, \tag{O1.1}$$

subject to the flow budget constraint $n' = R'_{i,n}(1-c_i)$, the natural borrowing limit $n' \geq 0$, and the portfolio-return constraint

$$\mathbb{E}_s[\Lambda' R'_{i,n}] = 1. \tag{O1.2}$$

ii. Consumption-wealth ratio and the investor's SDF are given by

$$c_i(X,s) = \frac{(\beta^{-1}-1)^{\psi} \mathcal{R}_i(X,s)^{1-\psi}}{1 + (\beta^{-1}-1)^{\psi} \mathcal{R}_i(X,s)^{1-\psi}}, \tag{O1.3}$$

$$\Lambda_i(X,s,s') = \beta^{\theta} \left(\frac{c_i(\chi(X,s,s'), s') N'}{c_i(X,s) N} \right)^{-\frac{\theta}{\psi}} R_{i,n}(X,s,s')^{-(1-\theta)}, \tag{O1.4}$$

and the change-of-measure condition is given by

$$\Lambda_i(X,s,s') = \frac{p_{ss'}}{p^i_{ss'}}\Lambda(X,s,s'). \tag{O1.5}$$

iii. *Wages, hours, and profits are given by*

$$h(E) = \left(\frac{\alpha E}{\xi}\right)^{\frac{1}{1+\nu-\alpha}}, \qquad w(E) = \xi\left(\frac{\alpha E}{\xi}\right)^{\frac{\nu}{1+\nu-\alpha}}, \qquad \pi(E,s) = \left(\frac{\alpha}{\xi}\right)^{\frac{\alpha}{1+\nu-\alpha}}\left[x_s E^{\frac{\alpha}{1+\nu-\alpha}} - \alpha E^{\frac{1+\nu}{1+\nu-\alpha}}\right]. \tag{O1.6}$$

iv. *The law of motion of the endogenous aggregate state variables is given by*

$$E'(X,s) = \sum_{s'\in\mathcal{S}} \frac{p_{ss'}\Lambda(X,s,s')}{\sum_{\tilde{s}\in\mathcal{S}} p_{s\tilde{s}}\Lambda(X,s,\tilde{s})} x_{s'}, \tag{O1.7}$$

$$\eta_i'(X,s,s') = \frac{(1-\kappa)\eta_i R_{i,n}(X,s,s')(1-c_i(X,s))}{\sum_{j=1}^{I}\eta_j R_{j,n}(X,s,s')(1-c_j(X,s))} + \kappa\mu_i. \tag{O1.8}$$

v. *The market clearing conditions for consumption and the Arrow security for state $s \in \mathcal{S}$ are given by*

$$\sum_{i=1}^{I}\eta_i c_i(X,s) = \frac{x_s h(E)^\alpha - \xi\frac{h(E)^{1+\nu}}{1+\nu}}{P(X,s)}, \qquad \sum_{i=1}^{I}\tilde{\eta}_i R_{n,i}(X,s,s') = R_p(X,s,s'), \tag{O1.9}$$

where $\tilde{\eta}_i \equiv \frac{\eta_i(1-c_i(X,s)}{\sum_{j=1}^{I}\eta_j(1-c_j(X,s)}$.

Proof. See Online Appendix O3.1. □

An implication of the result above is that the LDF corresponds to the risk-neutral expectation of productivity growth. The following corollary shows that $E'(X,s)$ can be expressed as the expected productivity growth (under the objective probability measure) discounted by a risk premium.

Corollary 4. *Let $R_g(X,s,s')$ denote the return on a claim on productivity growth, then*

$$\log E'(X,s) = \log \mathbb{E}_s[x_{s'}] - \log \overline{R}^e_g(X,s), \tag{O1.10}$$

where $\overline{R}^e_g(X,s) \equiv \sum_{s'\in\mathcal{S}} p_{ss'}\frac{R_g(X,s,s')}{R_b(X,s)}$ *is the risk premium on a claim on productivity growth.*

Proof. The price of a claim on productivity growth is given by

$$P_g(X,s) = \mathbb{E}_s[\Lambda(X,s,s')x_{s'}], \tag{O1.11}$$

and the return on this claim is given by $R_g(X,s,s') = \frac{x_{s'}}{P_g(X,s)}$.

Expressing the pricing condition above in terms of covariances, we obtain

$$\mathbb{E}_s[R_g(X,s,s')] - R_b(X,s) = -Cov_s\left(\frac{\Lambda(X,s,s')}{\mathbb{E}_s[\Lambda(X,s,s')]}, \frac{x_{s'}}{P_g(X,s)}\right). \tag{O1.12}$$

Similarly, we can write $E'(X,s)$ in terms of covariances:

$$E'(X,s) = \mathbb{E}_s[x_{s'}] + Cov_s\left(\frac{\Lambda(X,s,s')}{\mathbb{E}_s[\Lambda(X,s,s')]}, x_{s'}\right). \tag{O1.13}$$

Using the fact that $P_g(X,s) = E'(X,s)/R_b(X,s)$, we can combine the expressions above to obtain

$$E'(X,s) = \mathbb{E}_s[x_{s'}] - \left(\frac{\mathbb{E}_s[R_g(X,s,s')]}{R_b(X,s)} - 1\right)E'(X,s) \Rightarrow E'(X,s) = \frac{\mathbb{E}_s[x_{s'}]}{\overline{R}^e_g(X,s)}. \tag{O1.14}$$

□

O1.2 Special Case I: Log utility

We consider next the special case where $\psi = \gamma = 1$ for the economy with an arbitrary number of states. Proposition O.2 below shows that the main implications from Section 4 extends to this more general economy.

Proposition O.2 (Log-utility)**.** *Suppose $\psi = \gamma = 1$ and that the following condition is satisfied $x^N < \frac{x^1}{\alpha}$.*

i. Consumption and portfolio decisions are given by

$$c_i(X,s) = 1-\beta, \qquad R_{i,n}(X,s,s') = \frac{p^i_{ss'}}{p_{ss'}(X)}R_p(X,s,s'). \tag{O1.15}$$

ii. The economy's SDF is given by

$$\Lambda(X,s,s') = \frac{p_{ss'(X)}}{p_{ss'}}R'_p(X,s,s')^{-1}. \tag{O1.16}$$

iii. The price and return on the surplus claim are given by

$$P(X,s) = \frac{x_s h(E)^\alpha - \xi \frac{h(E)^{1+\nu}}{1+\nu}}{1-\beta}, \tag{O1.17}$$

$$R_p(X,s,s') = \frac{x_s}{\beta} \frac{x_{s'} E'(X,s)^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu} E'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}}}{x_s E^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu} E^{\frac{1+\nu}{1+\nu-\alpha}}}. \tag{O1.18}$$

iv. The risk premium on the surplus claim and the interest rate are given by

$$R_b(X,s) = \left(1 - \frac{\alpha}{1+\nu}\right) \frac{x_s}{\beta} \frac{E'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}}}{x_s E^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu} E^{\frac{1+\nu}{1+\nu-\alpha}}}, \tag{O1.19}$$

$$\mathbb{E}_s[R_p^e(X,s,s')] = \frac{x_s}{\beta} \frac{[\mathbb{E}_s[x_{s'}] - E(X,s)] E'(X,s)^{\frac{\alpha}{1+\nu-\alpha}}}{x_s E^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu} E^{\frac{1+\nu}{1+\nu-\alpha}}}. \tag{O1.20}$$

v. The law of motion of the aggregate state variables are given by

$$E'(X,s) = \sum_{s' \in \mathcal{S}} x_{s'} \frac{p_{ss'}(X) \left[x_{s'} - \frac{\alpha}{1+\nu} E'(X,s)\right]^{-1}}{\sum_{\tilde{s} \in \mathcal{S}} p_{s\tilde{s}}(X) \left[x_{\tilde{s}} - \frac{\alpha}{1+\nu} E'(X,s)\right]^{-1}} \tag{O1.21}$$

$$\eta_i'(X,s,s') = (1-\kappa)\eta_i \frac{p_{ss'}}{p_{ss'}(X)} + \kappa \mu_i, \tag{O1.22}$$

and there exists a unique value of $E'(X,s) \in (x_1, x_N)$ satisfying the law of motion of $\mathcal{L}$.

Proof. See Online Appendix O3.2. □

O1.3 Special Case II: Representative Agent with IID Returns

We consider next a different special case which is also particularly tractable: investors have common iid beliefs, $p^i_{ss'} = p^*_{s'}$, and the supply and demand of labor converge to zero. Formally, we assume $\alpha = \hat{\alpha}\epsilon$ and $\xi = \hat{\xi}\epsilon$ and take the limit as ϵ goes to zero. For simplicity, we focus on the case $\kappa = 0$. Because labor is chosen in advance, returns on financial assets would not be iid even if the process for aggregate productivity is iid. By taking the limit as supply and demand goes to zero, we ensure that all equilibrium objects are well-defined in the limit and the economy behaves essentially as an endowment economy, analogous to an iid version of the Mehra and Prescott (1985) economy.

Proposition O.3 provides a characterization of this limit economy. To highlight these results apply to this particular limit, we denote the equilibrium objects in the limiting economy with an $*$, e.g. $v^*(X,s)$ and $c^*(X,s)$.

Proposition O.3 (IID Returns). *Suppose $p^i_{ss'} = p^*_{s'}$, $\alpha = \hat{\alpha}\epsilon$, and $\xi = \hat{\xi}\epsilon$. Suppose also the following condition is satisfied: $\beta^* \equiv \beta \mathbb{E}^*[x_{s'}^{1-\gamma}]^{\frac{1-\psi^{-1}}{1-\gamma}} < 1$. Then, the economy in the limit as $\epsilon \to 0$ satisfies the following conditions:*

i. Consumption and portfolio decisions:

$$c_i^*(X,s) = 1 - \beta^*, \qquad R^*_{i,n}(X,s,s') = R^*_p(X,s,s'). \tag{O1.23}$$

ii. The net-worth multiplier $v_i^*(X,s)$ is given

$$v_i^*(X,s) = (1-\beta)^{\frac{1}{1-\psi^{-1}}}(1-\beta^*)^{-\frac{\psi^{-1}}{1-\psi^{-1}}}. \tag{O1.24}$$

iii. Wages, hours, and profits are given by

$$h^*(E) = \left(\frac{\hat{\alpha}E}{\hat{\xi}}\right)^{\frac{1}{1+\nu}}, \qquad w^*(E) = 0, \qquad \pi^*(E,s) = x_s. \tag{O1.25}$$

iv. The economy's SDF is given by

$$\Lambda^*(X,s,s') = \beta \mathbb{E}^*[x_{s'}^{1-\gamma}]^{\frac{\gamma-\psi^{-1}}{1-\gamma}} x_{s'}^{-\gamma}. \tag{O1.26}$$

v. The price and return on the surplus claim are given by

$$P^*(X,s) = \frac{x_s}{1-\beta^*}, \qquad R^*_p(X,s,s') = \frac{x'_s}{\beta^*}. \tag{O1.27}$$

vi. The risk-free rate and the expected return on the surplus claim are given by

$$R^*_b(X,s) = \frac{1}{\beta^*}\frac{\mathbb{E}^*[x_{s'}^{1-\gamma}]}{\mathbb{E}^*[x_{s'}^{-\gamma}]}, \tag{O1.28}$$

$$\mathbb{E}^*[R_p(X,s,s')] = R_b(X,s)\frac{\mathbb{E}^*[x_{s'}]\mathbb{E}^*[x_{s'}^{-\gamma}]}{\mathbb{E}^*[x_{s'}^{1-\gamma}]}. \tag{O1.29}$$

vii. The law of motion of the state variables are given by

$$E'(X,s) = E^*, \qquad \eta'_i(X,s,s') = \eta_i. \tag{O1.30}$$

where $E^* \equiv \frac{\mathbb{E}^*[x_{s'}^{1-\gamma}]}{\mathbb{E}^*[x_{s'}^{-\gamma}]}$.

Proof. See Online Appendix O3.3. □

The following corollary shows that we recover the standard asset pricing formulae for iid economies in continuous time if we assume that x_s is approximately log-normal.

Corollary 5. *Suppose* $\log x_s$ *can be approximated by a normal distribution with mean* μ *and variance* σ^2*. Then, under the assumption of Proposition O.3, we obtain*

i. Interest rate:

$$\log R_b^*(X,s) \approx \rho + \psi^{-1}\left(\mu + \frac{\sigma^2}{2}\right) - \frac{\gamma(1+\psi^{-1})}{2}\sigma^2, \tag{O1.31}$$

where $\rho \equiv -\log\beta$.

ii. Risk-premium:

$$\log \mathbb{E}\left[\frac{R_p^*(X,s,s')}{R_b^*(X,s)}\right] \approx \gamma\sigma^2. \tag{O1.32}$$

iii. Risk-neutral expectation of productivity growth:

$$\log \frac{E'(X,s)}{\mathbb{E}[x_{s'}]} \approx -\gamma\sigma^2. \tag{O1.33}$$

The corollary above shows how $E'(X,s)$ depends on $x_{s'}$ and the equity risk premium.

O2 Approximate Solution of the General Economy

In the previous section, we derived exact analytical solutions for two special cases: i) log-utility; ii) homogeneous beliefs and iid returns. In this section, we derive asymptotic closed-form solutions for a general economy with an arbitrary number of states, an arbitrary number of households with heterogeneous beliefs, and Epstein-Zin preferences with unrestricted EIS and risk aversion. The derivations for the benchmark case with homogeneous beliefs and iid returns will be useful in deriving the approximate solution.

O2.1 Perturbation

Consider a family of economies indexed by ϵ. The parameter ϵ controls three dimensions through which these economies differ from each other. First, it determines the degree of belief heterogeneity:

$$p^i_{ss'} = p^*_{s'} + \delta^i_{ss'}\epsilon, \tag{O2.1}$$

where $\sum_{s'\in\mathcal{S}} \delta^i_{ss'} = 0$. We also assume that the objective measure coincides with beliefs in the reference economy, i.e. $p_{ss'} = p^*_{s'}$. Second, ϵ scales both supply and demand for labor:

$$\xi = \hat{\xi}\epsilon, \qquad\qquad \alpha = \hat{\alpha}\epsilon. \tag{O2.2}$$

The economy satisfying $\epsilon = 0$ is essentially an endowment economy with iid common beliefs, a special case of the Mehra and Prescott (1985) economy, as described above. Third, we assume that $\kappa = \hat{\kappa}\epsilon$, such that there is no mortality risk in the benchmark economy.

All equilibrium objects are now indexed by ϵ. For instance, the net worth multiplier is now given by $v_i(X, s; \epsilon)$. We are interested in the expansion of $v_i(X, s; \epsilon)$ on ϵ, for ϵ small:

$$v_i(X, s; \epsilon) = v^*_i(X, s) + \hat{v}_i(X, s)\epsilon + \mathcal{O}(\epsilon^2), \tag{O2.3}$$

where $v^*_i(X, s) \equiv v_i(X, s; 0)$ and $\hat{v}_i(X, s)$ represents the first-order correction of $v_i(X, s; \epsilon)$ in ϵ.

Similarly, we can write the consumption-wealth ratio $c_i(X, s; \epsilon)$ as follows:

$$c_i(X, s; \epsilon) = c^*_i(X, s) + \hat{c}_i(X, s)\epsilon + \mathcal{O}(\epsilon^2), \tag{O2.4}$$

and analogously for the remaining equilibrium variables.

The functions $v^*_i(X, s)$ and $c^*_i(X, s)$ are already known, as they correspond to the solution of the case with homogeneous beliefs and iid returns, which we characterized above. It remains to solve for $\hat{v}_i(X, s)$, $\hat{c}_i(X, s)$, and the first-order correction for the other variables.

We start by providing a characterization of the households' problem in this general economy.

Proposition O.4. *Suppose that $p^i_{ss'} = p^*_{s'} + \delta^i_{ss'}\epsilon$, $\alpha = \hat{\alpha}\epsilon$, and $\xi = \hat{\xi}\epsilon$. Suppose also that $\beta^* < 1$. Then,*

i. Net-worth multiplier:

$$\frac{\hat{v}_i(X, s)}{v^*_i(X, s)} = \beta^* \sum_{s'\in\mathcal{S}} \omega^*_{s'} \left[\frac{1}{1-\gamma} \frac{\delta^i_{ss'}}{p^*_{s'}} - \frac{\hat{\Lambda}(X, s, s')}{\Lambda^*(X, s, s')} \right] + \beta^* \overline{v}, \tag{O2.5}$$

where $\omega^*_{s'} \equiv \frac{p^*_{s'} x^{1-\gamma}_{s'}}{\sum_{\tilde{s} \in \mathcal{S}} p^*_{\tilde{s}} x^{1-\gamma}_{\tilde{s}}}$, $X^* = (E^*, \{\eta_i\}_{i=1}^I)$, *and*

$$\bar{v} \equiv \frac{\beta^*}{1-\beta^*} \sum_{\tilde{s} \in \mathcal{S}} \omega^*_{\tilde{s}} \sum_{\tilde{s}' \in \mathcal{S}} \omega^*_{\tilde{s}'} \left[\frac{1}{1-\gamma} \frac{\delta^i_{\tilde{s}\tilde{s}'}}{p^*_{\tilde{s}'}} - \frac{\hat{\Lambda}(X^*, \tilde{s}, \tilde{s}')}{\Lambda^*(X^*, \tilde{s}, \tilde{s}')} \right]. \tag{O2.6}$$

ii. Consumption-wealth ratio:

$$\frac{\hat{c}_i(X,s)}{c^*(X,s)} = (1-\psi) \frac{\hat{v}_i(X,s)}{\hat{v}^*(X,s)}. \tag{O2.7}$$

iii. Portfolio return:

$$\frac{\hat{R}_{n,i}(X,s,s')}{R^*_p(X,s,s')} = \frac{1}{\gamma} myopic + \frac{1-\gamma}{\gamma} hedging, \tag{O2.8}$$

where

$$myopic = \left[\frac{\delta^i_{ss'}}{p^*_{s'}} - \sum_{\tilde{s}' \in \mathcal{S}} \omega^*_{\tilde{s}'} \frac{\delta^i_{s\tilde{s}'}}{p^*_{\tilde{s}'}} \right] - \frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')} \tag{O2.9}$$

$$hedging = \left[\frac{\hat{v}_i(X^*,s')}{v^*(X,s)} - \sum_{\tilde{s} \in \mathcal{S}} \omega^*_{\tilde{s}} \frac{\hat{v}_i(X^*,\tilde{s})}{v^*(X,s)} \right] + \sum_{\tilde{s} \in \mathcal{S}} \omega^*_{\tilde{s}} \frac{\hat{\Lambda}(X,s,\tilde{s})}{\Lambda^*(X,s,\tilde{s})}. \tag{O2.10}$$

Proof. See Online Appendix O3.4. □

Proposition O.4 provides asymptotic closed-form solutions to the value function and policy functions. The net-worth multiplier $\hat{v}_i(X,s)$ is high when investor i is relatively optimistic and state-prices are relatively low. The effect of beliefs can be seen by writing the term involving $\delta^i_{ss'}$ as follows:

$$\sum_{s' \in \mathcal{S}} \omega^*_{s'} \frac{1}{1-\gamma} \frac{\delta^i_{ss'}}{p^*_{s'}} = \sum_{s' \in \mathcal{S}} p^*_{s'} \frac{1}{1-\gamma} \frac{x^{1-\gamma}_{s'}}{\mathbb{E}^*[x^{1-\gamma}_{s'}]} \frac{\delta^i_{ss'}}{p^*_{s'}} = Cov^* \left(\frac{1}{1-\gamma} \frac{x^{1-\gamma}_{s'}}{\mathbb{E}^*[x^{1-\gamma}_{s'}]}, \frac{\delta^i_{ss'}}{p^*_{s'}} \right), \tag{O2.11}$$

using the fact that $\sum_{s' \in \mathcal{S}} p^*_{s'} \frac{\delta^i_{ss'}}{p^*_{s'}} = 0$. The covariance above will be positive when $\delta^i_{ss'}$ is on average positive when $x_{s'}$ is high, i.e. the covariance is increasing in how optimistic investor i is.

The term involving $\hat{\Lambda}(X,s,s')$ captures the effect of changes in the SDF on the portfolio return that can be achieved by the household:

$$1 = \mathbb{E}_s \left[\Lambda(X,s,s';\epsilon) R_{i,n}(X,s,s';\epsilon) \right] \Rightarrow \sum_{s' \in \mathcal{S}} \omega^*_{s'} \frac{\hat{R}_{i,n}(X,s,s')}{R^*_{i,n}(X,s,s')} = - \sum_{s' \in \mathcal{S}} \omega^*_{s'} \frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')}. \tag{O2.12}$$

Hence, if $\sum_{s' \in \mathcal{S}} \omega^*_{s'} \frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')}$ is negative, then the household is able to achieve higher weighted portfolio returns in the $\epsilon > 0$ economy.

Given $\hat{v}_i(X,s)$, we can characterize the policy functions. The consumption-wealth ratio $\hat{c}_i(X,s)$ is proportional to $\hat{v}_i(X,s)$. If $\psi > 1$, such that the substitution effect on savings dominates, households save a larger fraction of their wealth when average portfolio returns are high.

As in the continuous-time model of Merton (1992), portfolio returns have two components: the *myopic demand* and the *hedging demand*. The myopic demand depends on current market conditions, while the hedging demand depends on future expected returns as captured by $\hat{v}_i(X^*, s')$.

We consider next the labor market outcomes and firms' profits.

Proposition O.5 (Hours, wages, and profits). *Suppose that $p^i_{ss'} = p^*_{s'} + \delta^i_{ss'}\epsilon$, $\alpha = \hat{\alpha}\epsilon$, and $\xi = \hat{\xi}\epsilon$. Suppose also that $\beta^* < 1$. Then,*

i. Wages:

$$\hat{w}(E) = \hat{\xi}\left(\frac{\hat{\alpha}E}{\hat{\xi}}\right)^{\frac{\nu}{1+\nu}}. \tag{O2.13}$$

ii. Hours:

$$\hat{h}(E) = \left(\frac{\hat{\alpha}E}{\hat{\xi}}\right)^{\frac{1}{1+\nu}} \frac{\log\left(\frac{\hat{\alpha}E}{\hat{\xi}}\right)}{(1+\nu)^2}\hat{\alpha}. \tag{O2.14}$$

iii. Profits:

$$\hat{\pi}(X,s) = \left[x_s \frac{\log\left(\hat{\alpha}E/\hat{\xi}\right)}{1+\nu} - E\right]\hat{\alpha}. \tag{O2.15}$$

Proof. See Online Appendix O3.5 □

We consider next the behavior of the price of the surplus claim and the riskless asset.

Proposition O.6 (Asset Prices). *Suppose that $p^i_{ss'} = p^*_{s'} + \delta^i_{ss'}\epsilon$, $\alpha = \hat{\alpha}\epsilon$, and $\xi = \hat{\xi}\epsilon$. Suppose also that $\beta^* < 1$. Then,*

i. Price of surplus claim:

$$\frac{\hat{P}(X,s)}{P^*(X,s)} = \left[\log(\hat{\alpha}E/\hat{\xi}) - \frac{E}{x_s}\right]\frac{\hat{\alpha}}{1+\nu} + (\psi - 1)\sum_{i=1}^{I}\eta_i \frac{\hat{v}_i(X,s)}{v^*(X,s)}. \tag{O2.16}$$

ii. Return on the surplus claim:

$$\frac{\hat{R}_p(X,s,s')}{R^*_p(X,s,s')} = \left[\log\frac{E^*}{E} - \left(\frac{E^*}{x_{s'}} - \frac{E}{x_s}\right)\right]\frac{\hat{\alpha}}{1+\nu} + (\psi-1)\sum_{i=1}^{I}\eta_i\left[\frac{\hat{v}_i(X^*,s')}{v^*(X^*,s')} - \frac{1}{\beta^*}\frac{\hat{v}_i(X,s)}{v^*(X,s)}\right]. \tag{O2.17}$$

iii. Risk-free rate:

$$\frac{\hat{R}_b(X,s)}{R^*_b(X,s)} = -\sum_{s' \in \mathcal{S}} \frac{p_{s'}x_{s'}^{-\gamma}}{\mathbb{E}^*[x_{s'}^{-\gamma}]}\frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')}. \tag{O2.18}$$

iv. Conditional risk premium:

$$\frac{\widehat{\overline{R}}_E(X,s)}{\overline{R}_E^*(X,s)} = \sum_{s' \in \mathcal{S}} \frac{p_{s'}^* x_{s'}}{\mathbb{E}^*[x_{s'}]} \frac{\hat{R}_E(X,s,s')}{R_E^*(X,s,s')} - \frac{\hat{R}_b(X,s)}{R_b^*(X,s)}, \tag{O2.19}$$

where $\overline{R}_E(X,s;\epsilon) = \mathbb{E}^*\left[\frac{R_E(X,s,s';\epsilon)}{R_b(X,s;\epsilon)}\right]$.

Proof. See Online Appendix O3.6 □

The next proposition provides the law of motion of the aggregate state variables.

Proposition O.7 (Aggregate state variables.)**.** *Suppose that* $p_{ss'}^i = p_{s'}^* + \delta_{ss'}^i \epsilon$*,* $\alpha = \hat{\alpha}\epsilon$*, and* $\xi = \hat{\xi}\epsilon$*. Suppose also that* $\beta^* < 1$*. Then,*

i. Wealth distribution:

$$\frac{\hat{\eta}_i'(X,s,s')}{\eta_i} = \frac{\hat{R}_{i,n}(X,s,s')}{R_{i,n}^*(X,s,s')} - \sum_{j=1}^{I} \eta_i \frac{\hat{R}_{j,n}(X,s,s')}{R_{j,n}^*(X,s,s')} - ((\beta^*)^{-1} - 1)\left(\frac{\hat{c}_i(X,s)}{c_i^*(X,s)} - \sum_{j=1}^{I} \eta_j \frac{\hat{c}_j(X,s)}{c_j^*(X,s)}\right) + \kappa \frac{\mu_i - \eta_i}{\eta_i}. \tag{O2.20}$$

ii. Risk-neutral expectation of productivity growth:

$$\frac{\hat{E}'(X,s)}{E^*} = \frac{\hat{R}_b(X,s)}{R_b^*(X,s)} + \sum_{s' \in \mathcal{S}} \omega_{s'}^* \frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')}. \tag{O2.21}$$

Proof. See Online Appendix O3.7. □

Propositions O.4 to O.7 characterize the behavior of all equilibrium objects given the economy's SDF $\hat{\Lambda}(X,s,s')$. The next proposition provides an expression for $\Lambda(X,s,s')$ in terms of the primitives of the economy.

Proposition O.8 (The economy's SDF)**.** *Suppose that* $p_{ss'}^i = p_{s'}^* + \delta_{ss'}^i \epsilon$*,* $\alpha = \hat{\alpha}\epsilon$*, and* $\xi = \hat{\xi}\epsilon$*. Suppose also that* $\beta^* < 1$*. Then, the economy's SDF is given by*

$$\frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')} = \gamma b^{\Lambda}(X,s,s') - (\gamma - \psi^{-1})\left[\omega^* b^{\Lambda}(X,s) - \beta^* \omega^* \cdot b^{\Lambda}(X^*,s') + \beta^* \sum_{\tilde{s} \in \mathcal{S}} \omega_{\tilde{s}}(\omega^* \cdot b^{\Lambda}(X^*,\tilde{s}))\right], \tag{O2.22}$$

where

$$b^{\Lambda}(X,s,s') = \frac{1}{\gamma}\frac{\delta_{ss'}(X)}{p_{s'}^*} - \frac{\psi - \gamma^{-1}}{\gamma - 1}\left[\omega^* \cdot \delta_s(X) - \beta^* \omega^* \cdot \delta_{s'}(X) + \beta^* \sum_{\tilde{s}} \omega_{\tilde{s}}^*(\omega^* \cdot \delta_{\tilde{s}}(X))\right] - \left[\log \frac{E^*}{E} - \left(\frac{E^*}{x_{s'}} - \frac{E}{x_s}\right)\right]\frac{\hat{\alpha}}{1+\nu}. \tag{O2.23}$$

Proof. See Online Appendix O3.8. □

A particularly simple special case is given by the case of CRRA preferences, i.e. $\gamma = \psi^{-1}$.

Corollary 6. *Suppose $\gamma = \psi^{-1}$. Then,*

i. SDF:

$$\frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')} = \frac{\delta_{ss'}(X)}{p^*_{s'}} - \gamma\left[\log\frac{E^*}{E} - \left(\frac{E^*}{x_{s'}} - \frac{E}{x_s}\right)\right]\frac{\hat{\alpha}}{1+\nu}. \tag{O2.24}$$

O2.2 Special cases

Consider the special case where $\delta^i_{ss'} = 0$ for all $i \in \mathcal{I}$ and $s, s' \in \mathcal{S}$. In this case, investors still have common iid beliefs, but returns will not be iid due to the fact that labor is chosen one period in advance.

In this case, the economy's is given by

$$\frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')} = \psi^{-1}\left[\log\frac{E}{E^*} + \frac{E^*}{x_{s'}} - \frac{E}{x_s}\right]\frac{\hat{\alpha}}{1+\nu} - (\gamma - \psi^{-1})(1-\beta^*)\sum_{\tilde{s}'}\omega^*_{\tilde{s}'}\left(\frac{E^*}{x_{\tilde{s}'}} - \frac{E^*}{x_{s'}}\right)\frac{\hat{\alpha}}{1+\nu}. \tag{O2.25}$$

The interest rate is given by

$$\frac{\hat{R}_b(X,s)}{R^*_b(X,s)} = -\sum_{s'\in\mathcal{S}}\frac{p_{s'}x_{s'}^{-\gamma}}{\mathbb{E}^*[x_{s'}^{-\gamma}]}\frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')}. \tag{O2.26}$$

O2.3 Conditional moments

Consider the conditional risk premium

$$R^e_p(X,s;\epsilon) \equiv \sum_{s'\in\mathcal{S}} p^*_{s'}\frac{R_p(X,s,s';\epsilon)}{R_b(X,s;\epsilon)}. \tag{O2.27}$$

Expanding the expression above in ϵ, we obtain

$$\frac{\hat{R}^e_p(X,s)}{R^{e,*}_p} = \sum_{s'\in\mathcal{S}}\frac{p^*_{s'}R^*_p(X,s,s')}{\sum_{\tilde{s}'\in\mathcal{S}}p^*_{\tilde{s}'}R^*_p(X,s,\tilde{s}')}\left[\frac{\hat{R}_p(X,s,s')}{R^*_p(X,s,s')} - \frac{\hat{R}_b(X,s)}{R^*_b(X,s)}\right] = \sum_{s'\in\mathcal{S}}\frac{p_{s'}x_{s'}}{\mathbb{E}^*[x_{s'}]}\frac{\hat{R}_p(X,s,s')}{R^*_p(X,s)} - \frac{\hat{R}_b(X,s')}{R^*_b(X,s)}. \tag{O2.28}$$

The conditional volatility of excess returns is given by

$$\sigma_p(X,s;\epsilon) \equiv \left[\sum_{s'\in\mathcal{S}} p_{s'}\left(R^e_p(X,s,s';\epsilon) - R^e_p(X,s;\epsilon)\right)^2\right]^{\frac{1}{2}}. \tag{O2.29}$$

Expanding the expression above in ϵ, we obtain

$$\frac{\hat{\sigma}_p(X,s)}{\sigma^*_p(X,s)} = \frac{1}{\sigma^*_p(X,s)^2}\sum_{s'\in\mathcal{S}} p_{s'}\left(R^{e,*}_p(X,s,s') - R^{e,*}_p(X,s)\right)\left(\hat{R}^e_p(X,s,s') - \hat{R}^e_p(X,s)\right), \tag{O2.30}$$

where $\hat{R}^e_p(X,s,s') = \hat{R}^{e,*}_p(X,s,s')\left(\frac{\hat{R}_p(X,s,s')}{R^*_p(X,s,s')} - \frac{\hat{R}_b(X,s)}{R^*_b(X,s)}\right)$.

O2.4 Stock prices in the log economy

Suppose $\psi = \gamma = 1$ and that investors have homogeneous iid beliefs, $\delta^i_{ss'} = 0$. The stock price satisfies the following recursion:

$$Q(X,s) = \sum_{s' \in \mathcal{S}} p_{ss'} \Lambda(X,s,s') \left[\pi(E'(X,s),s') + x_{s'} Q(X',s') \right], \tag{O2.31}$$

where $\pi(E,s) = \left(\frac{\alpha E}{\xi}\right)^{\frac{\alpha}{1+\nu-\alpha}} x_s \left[1 - \frac{\alpha}{x_s} E\right]$ and $p_{ss'}\Lambda(X,s,s') = p_{ss'}(X) \frac{\beta}{x_{s'}} \left(\frac{E}{E'(X,s)}\right)^{\frac{\alpha}{1+\nu-\alpha}} \frac{1+\nu-\alpha \frac{E}{x_s}}{1+\nu-\alpha \frac{E'(X,s)}{x_{s'}}}$.

Define the price-dividend ratio $q(X,s) = x_s \frac{Q(X,s)}{\pi(X,s)}$. The price-dividend ratio satisfies the recursion:

$$q(X,s) = \sum_{s' \in \mathcal{S}} p_{ss'} \Lambda(X,s,s') \left[\pi(E'(X,s),s') + x_{s'} Q(X',s') \right], \tag{O2.32}$$

We can then write the expression above as follows:

$$q(X,s) = \beta \sum_{s' \in \mathcal{S}} p_{ss'}(X) \frac{1+\nu-\alpha\frac{E}{x_s}}{1+\nu-\alpha\frac{E'(X,s)}{x_{s'}}} \frac{1-\alpha\frac{E'(X,s)}{x_{s'}}}{1-\alpha\frac{E}{x_s}} \left[1 + q(X',s')\right]. \tag{O2.33}$$

Let's assume that $\nu = \bar{\nu}\epsilon$. We can then write $q(X,s;\epsilon)$ as follows

$$q(X,s;\epsilon) = \sum_{k=0}^{\infty} q_k(X,s)\epsilon^k. \tag{O2.34}$$

Define $g(X,s,s';\epsilon) \equiv \frac{1+\nu-\alpha\frac{E}{x_s}}{1+\nu-\alpha\frac{E'(X,s)}{x_{s'}}} \frac{1-\alpha\frac{E'(X,s)}{x_{s'}}}{1-\alpha\frac{E}{x_s}}$. We can expand $g(X,s,s')$ as follows

$$g(X,s,s';\epsilon) = \sum_{k=0}^{\infty} g_k(X,s,s')\epsilon^k, \tag{O2.35}$$

where $g_0(X,s,s') = 1$ and, for $k > 0$, we obtain

$$g_k(X,s,s') = \frac{\alpha\bar{\nu}^k \left(\frac{E'(X,s)}{x_{s'}} - \frac{E}{x_s}\right)}{\left(1-\alpha\frac{E}{x_s}\right)\left(\alpha\frac{E'(X,s')}{x_{s'}} - 1\right)^k} \tag{O2.36}$$

This gives the following recursion for $q_k(X,s)$:

$$q_k(X,s) = \beta \sum_{s' \in \mathcal{S}} p_{ss'}(X) \left[g_k(X,s) + \sum_{j=0}^{k} g_j(X,s) q_{k-j}(X',s') \right]. \tag{O2.37}$$

Under our assumptions, the risk-neutral expectation of productivity growth is constant

$E'(X,s) = \overline{E}$. In this case, we can write the recursion

$$Q(\overline{X},s) = \beta\left(1-\frac{\alpha}{1+\nu}\frac{\overline{E}}{x_s}\right)\sum_{s'\in\mathcal{S}}\frac{p^*_{s'}x_{s'}}{x_{s'}-\frac{\alpha}{1+\nu}\overline{E}}\left[Q(\overline{X},s')+\left(\frac{\alpha\overline{E}}{\xi}\right)^{\frac{\alpha}{1+\nu-\alpha}}\left(1-\alpha\frac{\overline{E}}{x_{s'}}\right)\right]$$

Let $\tilde{Q}(X,s) \equiv \frac{Q(x,s)}{1-\frac{\alpha}{1+\nu}\frac{E'(X,s)}{x_s}}$, $\tilde{Q}(X) \equiv [\tilde{Q}(X,1),\ldots,\tilde{Q}(X,N)]'$, and $b^Q \equiv \beta\left(\frac{\alpha\overline{E}}{\xi}\right)^{\frac{\alpha}{1+\nu-\alpha}}\sum_{s'\in\mathcal{S}}p^*_{s'}\frac{x_{s'}-\alpha\overline{E}}{x_{s'}-\frac{\alpha}{1+\nu}\overline{E}}$. Then, we can write

$$\left[I-\beta\mathbb{1}_N(p^*)'\right]\tilde{Q}(\overline{X}) = b^Q\mathbb{1}_N, \tag{O2.38}$$

Inverting the matrix above, we obtain

$$Q(\overline{X},s) = \left[\frac{\beta}{1-\beta}\left(\frac{\alpha\overline{E}}{\xi}\right)^{\frac{\alpha}{1+\nu-\alpha}}\sum_{s'\in\mathcal{S}}p^*_{s'}\frac{x_{s'}-\alpha\overline{E}}{x_{s'}-\frac{\alpha}{1+\nu}\overline{E}}\right]\left(1-\frac{\alpha}{1+\nu}\frac{\overline{E}}{x_s}\right). \tag{O2.39}$$

The price-dividend ratio is given by

$$\frac{x_sQ(\overline{X},s)}{\pi(\overline{X},s)} = \frac{\beta}{1-\beta}\sum_{s'\in\mathcal{S}}p^*_{s'}\frac{x_{s'}-\alpha\overline{E}}{x_{s'}-\frac{\alpha}{1+\nu}\overline{E}}\frac{x_s-\frac{\alpha}{1+\nu}\overline{E}}{x_s-\alpha\overline{E}}. \tag{O2.40}$$

Equity returns are given by

$$R_E(X,s,s') = \frac{\pi(\overline{E},s')+x'Q(\overline{X},s')}{Q(\overline{X},s)} = \frac{x_{s'}}{\beta}\left[\frac{1-\beta}{\sum_{s'\in\mathcal{S}}p^*_{s'}\frac{x_{s'}-\alpha\overline{E}}{x_{s'}-\frac{\alpha}{1+\nu}\overline{E}}}\frac{1-\alpha\frac{\overline{E}}{x_{s'}}}{1-\frac{\alpha}{1+\nu}\frac{\overline{E}}{x_s}}+\beta\frac{1-\frac{\alpha}{1+\nu}\frac{\overline{E}}{x_{s'}}}{1-\frac{\alpha}{1+\nu}\frac{\overline{E}}{x_s}}\right]. \tag{O2.41}$$

Excess returns are given by

$$R^e_E(X,s,s') = a_Ex_{s'}+b_E, \tag{O2.42}$$

where

$$a_E \equiv \frac{1}{\left(1-\frac{\alpha}{1+\nu}\right)\overline{E}}\left(\frac{1-\beta}{\sum_{s'\in\mathcal{S}}p^*_{s'}\frac{x_{s'}-\alpha\overline{E}}{x_{s'}-\frac{\alpha}{1+\nu}\overline{E}}}+\beta\right) \tag{O2.43}$$

$$b_E \equiv \tag{O2.44}$$

The conditional risk premium is given by

$$R_E(\overline{X},s) = \frac{1}{\left(1-\frac{\alpha}{1+\nu}\right)\overline{E}}\left[\left(\frac{1-\beta}{\sum_{s'\in\mathcal{S}}p^*_{s'}\frac{x_{s'}-\alpha\overline{E}}{x_{s'}-\frac{\alpha}{1+\nu}\overline{E}}}+\beta\right)\mathbb{E}[x_{s'}]-\left(\frac{(1-\beta)(1+\nu)}{\sum_{s'\in\mathcal{S}}p^*_{s'}\frac{x_{s'}-\alpha\overline{E}}{x_{s'}-\frac{\alpha}{1+\nu}\overline{E}}}+\beta\right)\frac{\alpha}{1+\nu}\overline{E}\right]-1. \tag{O2.45}$$

We can write the expression above as follows:

$$R_E(\overline{X}, s) = \frac{1}{\left(1 - \frac{\alpha}{1+\nu}\right)\overline{E}}\left[\left(\frac{1-\beta}{1-\nu\sum_{s'\in\mathcal{S}}\frac{p^*_{s'}\frac{\alpha}{1+\nu}\overline{E}}{x_{s'}-\frac{\alpha}{1+\nu}\overline{E}}}+\beta\right)\mathbb{E}[x_{s'}] - \left(\frac{(1-\beta)(1+\nu)}{1-\nu\sum_{s'\in\mathcal{S}}\frac{p^*_{s'}\frac{\alpha}{1+\nu}\overline{E}}{x_{s'}-\frac{\alpha}{1+\nu}\overline{E}}}+\beta\right)\frac{\alpha}{1+\nu}\overline{E}\right] - 1. \tag{O2.46}$$

O2.5 Quantitative implications

Let z_t denote demeaned log productivity growth, which we assume follows an AR(1) process:

$$z_{t+1} = \rho z_t + \sigma\sqrt{1-\rho^2}\epsilon_{t+1}, \tag{O2.47}$$

where ϵ_{t+1} follows a standard normal distribution and it is serially uncorrelated. In levels, the (gross) productivity growth is given by $x_t = e^{\mu + z_t}$, where μ denotes average productivity growth.

We discretize the process above following the method of Rouwenhurst (1995). Let $\hat{z}_t$ denote the discretized variable taking values in the equally-spaced grid $\{z_1, \ldots, z_N\}$, where $z_i = -\overline{\psi} + \frac{2\overline{\psi}}{N-1}(i-1)$, so $z_1 = -\overline{\psi}$ and $z_N = \overline{\psi}$. We set $\overline{\psi} \equiv \sigma\sqrt{N-1}$, so we match the unconditional variance.

O2.6 A more general process for productivity growth

Discretization. The evolution of $\hat{x}_t$, under subjective beliefs, can be written in a convenient matrix form:

$$\begin{bmatrix}\hat{x}_{t+1}\\ \hat{z}_{i,t+1}\end{bmatrix} = \begin{bmatrix} w_{t+1} \\ 0\end{bmatrix} + \begin{bmatrix}\theta_i & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}\hat{x}_t \\ \hat{z}_{i,t}\end{bmatrix} + \begin{bmatrix}\sigma_{i,u} & 0\\ 0 & \sigma_v\end{bmatrix}\begin{bmatrix}u_{i,t+1}\\ v_{t+1}\end{bmatrix}, \tag{O2.48}$$

where $\hat{z}_{i,t} \equiv \mathbb{E}_{i,t}[\hat{x}_{t+1}] - \theta_i \hat{x}_t$. We recover objective beliefs in the special case $\theta_i = \sigma_v = 0$. As w_{t+1} follows a Markov chain, the process above corresponds to a Markov-switching vector autoregression (MS-VAR), with state-dependent conditional means. To discretize the system above, we adapt the methods of Gospodinov and Lkhagvasuren (2014), who extended the Rouwenhurst (1995) method to VARs, and Liu (2017), who proposed a discretization of univariate Markov-Switching models. The discretization provides a state space with dimension S for x_t, so $x_t \in \mathcal{X} = \{x^1, x^2, \ldots, x^S\}$, and transition probabilities $\{p^i_{ss'}\}$, for $s, s' \in \mathcal{S} = \{1, 2, \ldots, S\}$, that approximate the MS-VAR (O2.48). Notice that our discretization implies that the grid $\mathcal{X}$ is the same for all investors, so they agree on the state s, but they disagree on the transition probabilities $p^i_{ss'}$.

Let $\hat{x}_t \equiv \log x_t - \mu$ denote the demeaned log productivity growth. We assume that investor i

believes the process for $\hat{x}_t$ is given by

$$\hat{x}_{t+1} = \mathbb{E}_{i,t}[\hat{x}_{t+1}] + \sigma_{i,u} u_{i,t+1} \tag{O2.49}$$

$$\mathbb{E}_{i,t}[\hat{x}_{t+1}] = \theta_i \hat{x}_t + \sigma_v v_{i,t}, \tag{O2.50}$$

$v_{i,t} = \overline{v}_t + \tilde{v}_{i,t}$, where $u_{i,t}$ and v_t are mutually independent, serially uncorrelated, standard normal random variables. Notice that $u_{i,t+1}$ represents the period $t+1$ innovation according to investor i and v_t represents an *expectation shock*. We assume that this expectation shock is common across investors, so heterogeneity comes only from θ_i.

The presence of this expectation shock is important to quantitatively match the volatility of expectations in the data. To see the role of v_t, notice that the unconditional variance of $\hat{x}_{t+1}$ and $\mathbb{E}_t[\hat{x}_{t+1}]$ are given by

$$Var[\hat{x}_{t+1}] = \frac{\sigma_{i,u}^2 + \sigma_v^2}{1 - \theta_i^2}, \qquad Var[\mathbb{E}_t[\hat{x}_{t+1}]] = \frac{\theta_i^2 \sigma_{i,u}^2 + \sigma_v^2}{1 - \theta_i^2}. \tag{O2.51}$$

The fraction of total variance explained by movements in expectations is given by

$$\frac{Var[\mathbb{E}_t[\hat{x}_{t+1}]]}{Var[\hat{x}_{t+1}]} = \theta_i^2 + (1 - \theta_i^2) \frac{\sigma_v^2}{\sigma_{i,u}^2 + \sigma_v^2}. \tag{O2.52}$$

Hence, by adjusting σ_v, it is possible to obtain any value in the interval $[\theta_i^2, 1)$ for the fraction of variance explained by movements in expectations. In the special case $\sigma_v = 0$, we obtain an AR(1) process for $\hat{x}_{t+1}$, which achieves the lower bound of this interval.

Discretization. We discretize the process above using the generalization of the method of Rouwenhurst (1995) proposed by Gospodinov and Lkhagvasuren (2014). The method consists of mixing the distribution for independent AR(1) processes to approximate the distribution of a VAR(1) with uncorrelated shocks. Define $\hat{z}_t \equiv \mathbb{E}_{i,t}[\hat{x}_{i,t}] - \theta_i \hat{x}_t$, so we can write the system above in matrix form:

$$\begin{bmatrix} \hat{x}_{t+1} \\ \hat{z}_{t+1} \end{bmatrix} = \begin{bmatrix} \theta & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_t \\ \hat{z}_t \end{bmatrix} + \begin{bmatrix} \sigma_u & 0 \\ 0 & \sigma_v \end{bmatrix} \begin{bmatrix} u_{t+1} \\ v_{t+1} \end{bmatrix}, \tag{O2.53}$$

where we dropped the dependence on the investor i to ease notation. Given this representation, we can construct the discrete approximation following the three steps described below.

Step 1: grid construction. We construct the grids for $\hat{x}$ and $\hat{z}$ as in Rouwenhurst (1995). Let $\hat{x}(N_x, \sigma_x) = \{\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^{N_x}\}$ denote the grid for $\hat{x}$, where

$$\hat{x}^i = -\psi_x(N_x, \sigma_x) + 2\psi_x(N_x, \sigma_x) \frac{i-1}{N_x - 1}, \tag{O2.54}$$

$\psi_x(N_x, \sigma_x) \equiv \sigma_x \sqrt{N_x - 1}$, and σ_x denotes the unconditional standard-deviation for $\hat{x}_t$. Notice that the grid is equally spaced, $\overline{x}^1 = -\psi_x(N_x, \sigma_x)$, and $\overline{x}^{N_x} = \psi_x(N_x, \sigma_x)$. The grid for $\hat{z}$ is constructed analogously.

Step 2: transition matrix for independent AR(1). Let $\Pi(N, \rho, \sigma)$ denote the $N \times N$ transition matrix for the Rouwenhurst (1995) approximation of an AR(1) process with autocorrelation ρ and unconditional variance σ^2. We denote the k-th row of this matrix by

$$\pi_k(N, \rho, \sigma) = \{\pi_{k,1}(N, \rho, \sigma), \pi_{k,2}(N, \rho, \sigma), \dots, \pi_{k,N}(N, \rho, \sigma)\}, \tag{O2.55}$$

where $\pi_{k,l}(N, \rho, \sigma)$ is the probability of transitioning from state k to state l. In the special case where $\rho = 0$, the transition probability is independent of the current state, so we can write $\pi_{k,j}(N, 0, \sigma) = \overline{\pi}_j(N, \sigma)$.

Step 3: Markov chain construction. Given N_x points in the grid for $\hat{x}$ and N_z points in the grid for $\hat{z}_t$, we have a total of $S = N_x \times N_z$ states. Denote the state space by $\mathcal{S} = \{1, 2, \dots, S\}$. Let $s = i + (k-1) \times N_x$ and $s' = j + (l-1) \times N_x$, where $i, j \in \{1, \dots, N_x\}$ and $k, l \in \{1, \dots, N_z\}$. Denote the probability of $\hat{x}_{t+1} = \overline{x}^j$ given state s by $p_s^x(j)$ and the probability of $\hat{z}_{t+1} = \overline{z}^l$ given state s by $p_s^z(l)$. As $\hat{x}_{t+1}$ and $\hat{z}_{t+1}$ are conditionally independent, the probability of switching from state s to state s' is given by

$$p_{ss'} = p_s^x(j) \times p_s^z(l). \tag{O2.56}$$

As z is serially uncorrelated, we have that $p_s^z(l) = \overline{\pi}_l(N_z, \sigma_z)$. The transition probability for $\hat{x}_t$ will be obtained by appropriately mixing the distribution of an AR(1) process with autocorrelation $\rho_x \equiv \sqrt{1 - \frac{\sigma_u^2}{\sigma_x^2}}$ and unconditional variance σ_x^2.

Let $\mu_x(s) \equiv \theta \hat{x}^i + \hat{z}^k$ denote the conditional expectation of $\hat{x}$ at state s. Suppose first that $\mu_x(s) \in [\rho_x \overline{x}^1, \rho_x \overline{x}^{N_x}]$. Define the probability of $\hat{x}_{t+1} = \overline{x}^j$ given state s as follows:

$$p_s^x(j) = \lambda(\rho_x)\pi_{\iota,j}(N_x, \rho_x, \sigma_x) + (1 - \lambda(\rho_x))\pi_{\iota+1,j}(N_x, \rho_x, \sigma_x), \tag{O2.57}$$

where ι is such that $\rho_x \overline{x}^\iota \leq \mu_x(s) \leq \rho_x \overline{x}^{\iota+1}$ and $\lambda(\rho_x) \equiv \frac{\rho_x \overline{x}^{\iota+1} - \mu_x(s)}{\rho_x \overline{x}^{\iota+1} - \rho_x \overline{x}^\iota}$.

This choice of $\lambda(\rho_x)$ implies that we match the conditional moments:

$$\sum_{j=1}^{N_x} p_s^x(j)\overline{x}^j = \lambda(\rho_x)\rho_x \overline{x}^\iota + (1 - \lambda(\rho_x))\rho_x \overline{x}^{\iota+1} = \mu_x(s). \tag{O2.58}$$

The conditional second moment is given by

$$\sum_{j=1}^{N_x} p_s^x(j)(\overline{x}^j)^2 = \sigma_x^2(1 - \rho_x^2) + \rho_x^2 \left[\lambda(\rho_x)(\overline{y}^\iota)^2 + (1 - \lambda(\rho_x))(\overline{y}^{\iota+1})^2\right]. \tag{O2.59}$$

Denote the conditional variance of the discrete process by $\tilde{\sigma}_u^2$, which is given by

$$\begin{aligned}\tilde{\sigma}_u^2 &= \sigma_x^2(1-\rho_x^2) + \rho_x^2\left[\lambda(\rho_x)(\overline{y}^{\iota})^2 + (1-\lambda(\rho_x))(\overline{y}^{\iota+1})^2\right] - \rho_x^2\left[\lambda(\rho_x)\overline{y}^{\iota} + (1-\lambda(\rho_x))\overline{y}^{\iota+1}\right]^2 \\ &= \sigma_x^2(1-\rho_x^2) + \rho_x^2\lambda(\rho_x)(1-\lambda(\rho_x))(\overline{x}^{\iota+1}-\overline{x}^{\iota})^2 \\ &= \sigma_x^2(1-\rho_x^2) + \sigma_x^2\rho_x^2\frac{4\lambda(\rho_x)(1-\lambda(\rho_x))}{N_x-1}, \end{aligned} \tag{O2.60}$$

using the fact that $(\overline{x}^{i+1}-\overline{x}^{i})^2 = \frac{4\sigma_x^2}{N-1}$. As $N_x \to \infty$, the second term on the right converges to zero and $\tilde{\sigma}_u = \sigma_x^2(1-\rho_x^2) = \sigma_u^2$, given our choice of ρ_x. If $\mu_x(s)/\rho_x$ does not belongs to the grid of $\hat{x}$, then the discretization matches the conditional mean of $\hat{x}$, but it overstates the conditional variance.

Suppose now that $\mu_x(s) \notin [\rho_x\overline{x}^1, \rho_x\overline{x}^{N_x}]$. In this case, we set $p_s^x(j) = \pi_{1,j}(N_x,\rho_x,\sigma_x)$ if $\mu_x(s) < \rho_x\overline{x}^1$ and $p_s^x(j) = \pi_{N_x,j}(N_x,\rho_x,\sigma_x)$ if $\mu_x(s) > \rho_x\overline{x}^{N_x}$. In both cases, the conditional variance is matched exactly and the conditional mean achieves the value closest to $\mu_x(s)$ given the grid points.

A different representation. An equivalent representation of the system is given by

$$\hat{x}_{t+1} = z_t + \sigma_u u_{t+1} \tag{O2.61}$$

$$z_{t+1} = \theta_i z_t + \theta_i\sigma_u u_{t+1} + \sigma_v v_{t+1}, \tag{O2.62}$$

where $z_t \equiv \mathbb{E}_{i,t}[\hat{x}_{t+1}]$. Hence, expected growth follows an AR(1) process and it is exposed to both expectation shocks, v_{t+1}, and shocks to realized growth rates, u_{t+1}. Notice that we cannot independently choose the persistence of expectations and the correlation between z_{t+1} and $\hat{x}_{t+1}$.

The impact of v_t in expected future growth is

$$\frac{\partial\mathbb{E}_t[\hat{x}_{t+k}]}{\partial v_t} = \sigma_v\theta_i^{k-1}, \tag{O2.63}$$

for $k \geq 1$.

O2.7 A process with richer heterogeneity

Under the objective measure, log productivity follows a Markov-Switching process:

$$\log(x_{t+1}) = \mu_{t+1} + \theta(\log(x_t) - \mu_t) + u_{t+1}, \tag{O2.64}$$

where $u_{t+1} \sim \mathcal{N}(0,\sigma_u^2)$ and μ_{t+1} follows a two-state Markov chain, that is, $\mu_{t+1} \in \{\mu^1,\mu^2\}$ and $Pr(\mu_{t+1} = \mu^j|\mu_t = \mu^i) = p_{ij}^{\mu}$ for $i,j \in \{1,2\}$. The different regimes enable us to capture the fact that productivity is subject to small fluctuations most of the time with occasional rare large shocks.

Under subjective beliefs, productivity follows the process

$$\log(x_{t+1}) = \mu_{t+1} + \theta_i(\log(x_t) - \mu_t) + v_{i,t} + u_{i,t+1}, \tag{O2.65}$$

where $u_{i,t+1} \sim \mathcal{N}(0, \sigma_{i,u}^2)$, $v_{i,t} = \rho\sigma_{i,v}\overline{v}_t + \sqrt{1-\rho^2}\sigma_{i,v}\hat{v}_{i,t}$, and $(\overline{v}_t, v_{i,t})$ are iid standard normal random variables. We assume that $(u_{i,t}, \hat{v}_{i,t}, \overline{v}_t)$ are mutually independent.

Subjective beliefs differ from the objective one in two important dimensions. First, the persistent parameter θ_i may differ from the objective one θ. Second, subjective beliefs are exposed to expectation shocks $v_{i,t}$. These expectations shocks are exposed to a common component $\overline{v}_t$ and an investor-specific component $v_{i,t}$. Differences in θ_i capture the fact that investors differ on how they react to news, with some investors extrapolating and some investors under-reacting. The expectation shocks $v_{i,t}$ are important to capture the volatility of subjective expectations observed in the data.

Define $\hat{x}_t \equiv \log(x_t) - \mu_t$ and the vector $\hat{v}_t = [\hat{v}_{1,t}, \ldots, \hat{v}_{I,t}]'$. Investor i believes that $[\hat{x}_t, \overline{v}_t, \hat{v}_t]$ follows the process:

$$\begin{bmatrix} \hat{x}_{t+1} \\ \overline{v}_{t+1} \\ \hat{v}_{t+1} \end{bmatrix} = \begin{bmatrix} \theta_i & \rho\sigma_{i,v} & \sqrt{1-\rho^2}\sigma_{i,v}\mathbf{e}_i' \\ 0 & 0 & \mathbf{0}_{1\times I} \\ \mathbf{0}_{I\times 1} & \mathbf{0}_{I\times 1} & \mathbf{0}_{I\times I} \end{bmatrix} \begin{bmatrix} \hat{x}_t \\ \overline{v}_t \\ \hat{v}_t \end{bmatrix} + \begin{bmatrix} u_{i,t+1} \\ \overline{v}_{t+1} \\ \hat{v}_{t+1} \end{bmatrix} \tag{O2.66}$$

Notice that the total variance and the variance of the conditional expectation are given by

$$Var[\hat{x}_{t+1}] = \frac{\sigma_{i,u}^2 + \sigma_{i,v}^2}{1-\theta_i^2}, \qquad Var[\mathbb{E}_t[\hat{x}_{t+1}]] = \frac{\theta_i^2\sigma_{i,u}^2 + \sigma_{i,v}^2}{1-\theta_i^2}. \tag{O2.67}$$

The fraction of total variance explained by movements in expectations is given by

$$\frac{Var[\mathbb{E}_t[\hat{x}_{t+1}]]}{Var[\hat{x}_{t+1}]} = \theta_i^2 + (1-\theta_i^2)\frac{\sigma_{i,v}^2}{\sigma_{i,u}^2 + \sigma_{i,v}^2}. \tag{O2.68}$$

O2.8 A more general process for productivity growth

Let $\hat{x}_t \equiv \log x_t - \mu$ denote the demeaned log productivity growth. We assume that $\hat{x}_t$ follows the process:

$$\hat{x}_{t+1} = z_t + \sigma_x\left[\sqrt{1-\rho_{xz}^2}u_{t+1} + \rho_{xz}v_{t+1}\right] \tag{O2.69}$$

$$z_{t+1} = \theta_z z_t + \sigma_z v_{t+1}, \tag{O2.70}$$

where u_t and v_t are standard normal random variables, serially uncorrelated, and uncorrelated with each other. Notice that $\mathbb{E}_t[x_{t+1}] = z_t$, so z_t corresponds to expected productivity growth. The disturbance v_{t+1} can then be interpreted as expectations shocks. These expectations shocks are potentially correlated with cash-flow shocks, with correlation coefficient ρ_{xz}. In the long-run risk literature, v_{t+1} is referred to as a long-run risk shock, while u_{t+1} corresponds to a short-run risk shock.

The ARMA(1,1) case. Suppose $\rho_{xz} = 1$. This implies that the process for $\hat{x}_t$ specializes to

$$\hat{x}_{t+1} = \theta_z \hat{x}_t + \sigma_x v_{t+1} - (\theta_z \sigma_x - \sigma_z) v_t, \tag{O2.71}$$

which is an ARMA(1,1) process. If we further assume that $\sigma_z = \theta_z \sigma_x$, then we obtain an AR(1) process.

Notice that we can write the conditional expectation of x_{t+1} as follows

$$\mathbb{E}_t[x_{t+1}] = \theta_z \hat{x}_t - b \frac{\hat{x}_t - \mathbb{E}_{t-1}[x_t]}{\sigma_x} \Rightarrow \mathbb{E}_t[x_{t+1}] = \frac{\theta_z - b/\sigma_x}{1 - bL} \hat{x}_t. \tag{O2.72}$$

where $b \equiv \theta_z \sigma_x - \sigma_z$ and L is the lag operator.

Define $\hat{w}_t$ as follows

$$\hat{w}_t \equiv \frac{\hat{x}_t}{1 - bL} = \sum_{j=1}^{\infty} b^j \hat{x}_{t-j}. \tag{O2.73}$$

Unconditional moments. The unconditional variance of z_{t+1} is given by

$$Var[z_{t+1}] = \frac{\sigma_z^2}{1 - \theta_z^2}, \tag{O2.74}$$

and the unconditional variance of $\hat{x}_{t+1}$ is given by

$$Var[\hat{x}_{t+1}] = \mathbb{E}\left[Var_t[\hat{x}_{t+1}]\right] + Var[\mathbb{E}_t[\hat{x}_{t+1}]] = \sigma_x^2 + \frac{\sigma_z^2}{1 - \theta_z^2}. \tag{O2.75}$$

In this general case, we can choose θ_z and σ_z to match the persistence and variance of expectations and choose σ_x^2 to match the unconditional variance of productivity growth. The parameter ρ_{xz} controls the correlation between expected and realized productivity growth.

In the special case $\rho_{xz} = 1$ and $\sigma_z = \theta_z \sigma_x$. This allows us to match the persistence of expectations and either the unconditional variance of expected productivity growth or unconditional variance of realized productivity growth.

If $\sigma_z = \theta_z \sigma_x$, then

$$Var[z_{t+1}] = \theta_z^2 \frac{\sigma_x^2}{1 - \theta_z^2} = \theta_z^2 Var[x_{t+1}]. \tag{O2.76}$$

Discretization of the productivity growth process. Using the process for z_{t+1} to eliminate v_{t+1} from the expression for $\hat{x}_t$, we obtain

$$\hat{x}_{t+1} = \theta_z^{-1} z_{t+1} + \left(\frac{\rho_{xz}\sigma_x}{\sigma_z} - \frac{1}{\theta_z} \right) (z_{t+1} - \theta_z z_t) + \sigma_x \sqrt{1-\rho_{xz}^2} u_{t+1}. \tag{O2.77}$$

Given z_t, z_{t+1}, and u_{t+1}, this allow us to solve for $\hat{x}_{t+1}$. Suppose z_t takes on N_z discrete values and u_t takes on N_u values. This implies that $\hat{x}_t$ can take on $N \equiv N_z^2 \times N_u$ values. If we impose the constraint $\rho_{xz} = 1$, then $\hat{x}_{t+1}$ is independent of u_{t+1}, so $\hat{x}_t$ takes on N_z^2 possible values. If we further assume that $\sigma_z = \theta_z \sigma_x$, then $\hat{x}_t$ can take only N_z values.

The current value of $\hat{x}$ is determined by $(z_{t-1}, z_t, u_t) = (z^i, z^j, u^k)$, where $i, j \in \{1, \ldots, N_z\}$ and $k \in \{1, \ldots, N_u\}$. We can define the current state as a function of (i, j, k): $s = i + (j-1)N_z + (k-1)N_z^2$. The transition matrix is then given by

$$Pr(s' = i' + (j'-1)N_z + (k'-1)N_z^2 | s) = \begin{cases} Pr(z' = z^{j'} | z = z^j) Pr(u' = u^k), & \text{if } i' = j \\ 0, & \text{if } i' \neq j \end{cases}, \tag{O2.78}$$

where $s = i + (j-1)N_z + (k-1)N_z^2$.

We can write the system above in matrix form:

$$\begin{bmatrix} \hat{x}_{t+1} \\ z_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & \theta_z \end{bmatrix} \begin{bmatrix} \hat{x}_t \\ z_t \end{bmatrix} + \begin{bmatrix} \sigma_x & 0 \\ \rho_{xz}\sigma_z & \sqrt{1-\rho_{xz}^2}\sigma_z \end{bmatrix} \begin{bmatrix} u_{t+1} \\ v_{t+1} \end{bmatrix}. \tag{O2.79}$$

Notice that the spectral decomposition of the matrix of coefficients is given by

$$\begin{bmatrix} 0 & 1 \\ 0 & \theta_z \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \theta_z & 0 \end{bmatrix} \begin{bmatrix} \theta_z & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \theta_z^{-1} \\ 1 & -\theta_z^{-1} \end{bmatrix}. \tag{O2.80}$$

Define the following transformed variables:

$$\begin{bmatrix} w_{1,t} \\ w_{2,t} \end{bmatrix} \equiv \begin{bmatrix} 0 & \theta_z^{-1} \\ 1 & -\theta_z^{-1} \end{bmatrix} \begin{bmatrix} x_t - \mu \\ z_t \end{bmatrix}. \tag{O2.81}$$

The difference equation for w_t is given by

$$\begin{bmatrix} w_{1,t+1} \\ w_{2,t+1} \end{bmatrix} = \begin{bmatrix} \theta_z & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_{1,t} \\ w_{2,t} \end{bmatrix} + \begin{bmatrix} \rho_{xz}\frac{\sigma_z}{\theta_z} & \sqrt{1-\rho_{xz}^2}\frac{\sigma_z}{\theta_z} \\ \sigma_x - \rho_{xz}\frac{\sigma_z}{\theta_z} & -\sqrt{1-\rho_{xz}^2}\frac{\sigma_z}{\theta_z} \end{bmatrix} \begin{bmatrix} u_{t+1} \\ v_{t+1} \end{bmatrix}. \tag{O2.82}$$

We can write the original variables in terms of $w_{1,t}$ and $w_{2,t}$:

$$\log x_t = \mu + w_{1,t} + w_{2,t}, \qquad z_t = \theta_z w_{1,t} \tag{O2.83}$$

We can then discretize $w_{1,t}$ and $w_{2,t}$.

$$w_{2,t+1} = \sqrt{1-\rho_{xz}^2}\sigma_x u_{t+1} + (\rho_{xz}\sigma_x - \sigma_z\theta_z^{-1})v_{t+1} \tag{O2.84}$$

$$w_{1,t+1} = \theta_z w_{1,t} + \sigma_z\theta_z^{-1}u_{t+1} \tag{O2.85}$$

$$x_{t+1} = \mu + w_{1,t+1} + \sqrt{1-\rho_{xz}^2}\sigma_x(w_{1,t+1} - \theta_z w_{1,t})\frac{\theta_z}{\sigma_z} + (\rho_{xz}\sigma_x - \sigma_z\theta_z^{-1})v_{t+1}. \tag{O2.86}$$

O3 Proofs

O3.1 Proof of Proposition O.1

Proof. We provide the characterization of the economy with N-possible states in steps, proceeding from the households' problem to the market clearing conditions.

Step 1: households' problem. Household i chooses consumption C_i, hours h_i, and arrow securities $B_i(X,s,s')$ to maximize (3) subject to the budget constraint:

$$C_i + \mathbb{E}_s[\Lambda(X,s,s')B_i(X,s,s')] = B_i + Wh_i, \tag{O3.1}$$

and an appropriate No-Ponzi condition.

As in the two-state case, it is useful to transform this budget constraint in terms of net consumption and total wealth:

$$\tilde{C}_i + \mathbb{E}_s\left[\Lambda'\left(B_i' + W'h_i' - \xi'\frac{(h_i')^{1+\nu}}{1+\nu} + \mathcal{H}_i'\right)\right] = B_i + Wh_i - \xi\frac{h_i^{1+\nu}}{1+\nu} + \mathcal{H}_i \equiv N_i, \tag{O3.2}$$

where we used the fact that $\mathcal{H}_i = \mathbb{E}_s\left[\Lambda'\left(W'h_i' - \xi'\frac{(h_i')^{1+\nu}}{1+\nu} + \mathcal{H}_i'\right)\right]$ and $\tilde{C}_i = C_i - \xi\frac{h_i^{1+\nu}}{1+\nu}$.

We can then write the budget constraint above as follows

$$\tilde{C}_i + \mathbb{E}_s\left[\Lambda' N_i'\right] = N_i. \tag{O3.3}$$

The household's problem can then be equivalently expressed as choosing $(\tilde{C}_i(N,X,s), N_i'(N,X,s,s'))$ to maximize (3) subject to the constraint above and the natural borrowing limit $N_i'(N,X,s,s') \geq 0$. The solution takes the form in Equation (12). It will be useful to define the consumption-wealth ratio $c_i \equiv \frac{\tilde{C}_i}{N}$ and the normalized net worth $n_i' \equiv \frac{N_i'}{N_i}$. Define the portfolio return as $R_{i,n}(X,s,s') \equiv \frac{n_i'(X,s,s')}{1-c_i(X,s)}$, which gives the budget constraint $n' = R_{i,n}'(1-c)$. The function $v_i(X,s)$ must then satisfy the condition

$$\frac{(v_i(X,s)N)^{1-\psi^{-1}}-1}{1-\psi^{-1}} = \max_{c_i,n_i'}(1-\beta)\frac{(c_iN)^{1-\psi^{-1}}-1}{1-\psi^{-1}} + \beta\frac{\mathbb{E}_i\left[(v_i(X',s')n'N)^{1-\gamma}\right]^{\frac{1-\psi^{-1}}{1-\gamma}}-1}{1-\psi^{-1}}, \quad \text{(O3.4)}$$

subject to $n' = R_n'(1-c_i)$, $\mathbb{E}_s[\Lambda' R_n'] = 1$, and $n' \geq 0$.

Step 2: optimality conditions. The first-order conditions for the consumption-wealth ratio and the portfolio share are given by

$$(1-\beta)c_i^{-\psi^{-1}} = \beta\mathcal{R}_i(X,s)^{1-\psi^{-1}}(1-c_i)^{-\psi^{-1}} \quad \text{(O3.5)}$$

$$p_{ss'}^i v_i(X',s')^{1-\gamma}R_{i,n}'(X,s,s')^{-\gamma} = p_{ss'}\Lambda(X,s,s')\mu(X,s), \quad \text{(O3.6)}$$

where $\mathcal{R}_i(X,s) = \mathbb{E}_i\left[(v_i(X',s')R_{i,n}(X,s,s'))^{1-\gamma}|X,s\right]^{\frac{1}{1-\gamma}}$ and $\mu(X,s)$ is the (normalized) multiplier on the constraint on returns. From the first-order condition for consumption, we obtain Equation (13). The envelope condition is given by

$$v_i(X)^{1-\frac{1}{\psi}} = (1-\beta)c_i^{-\frac{1}{\psi}} \Rightarrow v_i(X)^{1-\gamma} = (1-\beta)^{\theta}c_i^{-\frac{\theta}{\psi}}. \quad \text{(O3.7)}$$

Notice that the multiplier is given by

$$\mu(X,s) = \mathbb{E}_i[(v_i(X,s')R_{i,n}'(X,s,s'))^{1-\gamma}] = \mathcal{R}_i(X,s)^{1-\gamma} = \left(\frac{1-\beta}{\beta}\right)^{\theta}\left[\frac{c_i}{1-c_i}\right]^{-\frac{\theta}{\psi}}. \quad \text{(O3.8)}$$

Combining the previous two expressions above with the first-order condition for $R_{i,n}'$, we obtain

$$p_{ss'}\Lambda(X,s,s') = p_{ss'}^i\frac{(1-\beta)^{\theta}(c_i')^{-\frac{\theta}{\psi}}R_{i,n}'(X,s,s')^{-\gamma}}{\left(\frac{1-\beta}{\beta}\right)^{\theta}\left[\frac{c_i}{1-c_i}\right]^{-\frac{\theta}{\psi}}} \quad \text{(O3.9)}$$

$$= p_{ss'}^i\beta^{\theta}\left(\frac{c_i'N'}{c_iN}\right)^{-\frac{\theta}{\psi}}(R_{i,n}')^{-(1-\theta)}, \quad \text{(O3.10)}$$

$$\equiv p_{ss'}^i\Lambda_i(X,s,s'), \quad \text{(O3.11)}$$

using the fact that $\frac{\theta}{\psi}+1-\theta = \gamma$.

Hence, expressions (13) and (14) hold unchanged with multiple states. Moreover, the change-of-measure equation $\Lambda_i(X,s,s') = \frac{p_{ss'}}{p_{ss'}^i}\Lambda(X,s,s')$ also holds.

Step 3: firms' problem and labor market outcomes. The firm's problem is essentially the same and the first-order condition (19) holds without change. The equations for hours and wages

(21) are also unchanged.

Step 4: law of motion of aggregate state variables. The aggregate state variables are the same as before. The law of motion of $\mathcal{L}$ is given by

$$E'(X,s) = \sum_{s' \in \mathcal{S}} \frac{p_{ss'}\Lambda(X,s,s')}{\sum_{\tilde{s}\in\mathcal{S}} p_{s\tilde{s}}\Lambda(X,s,\tilde{s})} x_{s'}, \tag{O3.12}$$

and the law of motion of η_i is unchanged.

Step 5: market clearing conditions. Notice that $\sum_{i=1}^{I} \mu_i B_i$ must coincide with the cum-dividend value of the firm. Hence, $\sum_{i=1}^{I} \mu_i N_i$ coincides with the cum-dividend value of the surplus claim, $A_- P(X,s)$, where A_- denotes lagged productivity.

The market clearing condition for net consumption is then given by

$$\sum_{i=1}^{I} \mu_i N_i c_i(X,s) = A_- \left(x_s h(E)^\alpha - \xi \frac{h(E)^{1+\nu}}{1+\nu} \right). \tag{O3.13}$$

Using the fact that $\sum_{i=1}^{I} \mu_i N_i = A_- P(X,s)$, we obtain the market clearing for consumption in Equation (23). The market clearing for Arrow securities is given by

$$\sum_{i=1}^{I} \mu_i N_i n_i(X,s,s') = x_s A_- P(X',s'). \tag{O3.14}$$

We can write the expression above in terms of portfolio returns:

$$\sum_{i=1}^{I} \frac{\mu_i N_i (1-c_i(X,s))}{\sum_{j=1}^{I} \mu_j N_j (1-c_j(X,s))} R_{n,i}(X,s,s') = \frac{x_s A_- P(X',s')}{A_- \left[P(X,s) - \left(x_s h(E)^\alpha - \xi \frac{h(E)^{1+\nu}}{1+\nu} \right) \right]}, \tag{O3.15}$$

using the fact that $\sum_{j=1}^{I} \mu_j N_j (1-c_j(X,s)) = A_- \left[P(X,s) - \left(x_s h(E)^\alpha - \xi \frac{h(E)^{1+\nu}}{1+\nu} \right) \right]$.

We can write the expression above as follows

$$\sum_{i=1}^{I} \tilde{\eta}_i R_{n,i}(X,s,s') = R_p(X,s,s'), \tag{O3.16}$$

for each $s' \in \mathcal{S}$. □

O3.2 Proof of Proposition O.2

Proof. We provide next a characterization of the economy with log-utility and an arbitrary number of states.

Step 1: consumption and portfolio decisions. Suppose $\psi = \gamma = 1$. This implies that $c_i(X,s) = 1-\beta$ and that $\Lambda_i(X,s,s') = R_{i,n}^{-1}(X,s,s')$. From the change-of-measure equation, we obtain

$$R_{i,n}^{-1}(X,s,s') = \frac{p_{ss'}\Lambda(X,s,s')}{p_{ss'}^i} \Rightarrow R_{i,n}(X,s,s') = \frac{p_{ss'}^i}{p_{ss'}\Lambda(X,s,s')}. \tag{O3.17}$$

Step 2: the economy's SDF. Plugging the expression for $R_{i,n}(X,s,s')$ into the market clearing condition for Arrow securities paying off in state s', we obtain

$$\frac{\sum_{i=1}^{I}\eta_i p_{ss'}^i}{p_{ss'}\Lambda(X,s,s')} = R_p(X,s,s') \Rightarrow \Lambda(X,s,s') = \frac{p_{ss'}(X)}{p_{ss'}}R_p^{-1}(X,s,s'). \tag{O3.18}$$

Notice that the portfolio return for household i is given by

$$R_{i,n}(X,s,s') = \frac{p_{ss'}^i}{p_{ss'}(X)}R_p(X,s,s'). \tag{O3.19}$$

Hence, optimistic investors, i.e. investors satisfying $p_{ss'}^i > p_{ss'}(X)$, hold a levered position on the surplus claim.

Step 3: the surplus claim. From the market clearing condition for goods, we obtain

$$P(X,s) = \frac{x_s h(E)^\alpha - \xi\frac{h(E)^{1+\nu}}{1+\nu}}{1-\beta}. \tag{O3.20}$$

This implies that the return on the surplus claim is given by

$$R_p(X,s,s') = \frac{x_s P(X',s')}{P(X,s) - \left(x_s h(E)^\alpha - \xi\frac{h(E)^{1+\nu}}{1+\nu}\right)} = \frac{x_s}{\beta}\frac{x_{s'}h(E'(X,s))^\alpha - \xi\frac{h(E'(X,s))^{1+\nu}}{1+\nu}}{x_s h(E)^\alpha - \xi\frac{h(E)^{1+\nu}}{1+\nu}}. \tag{O3.21}$$

Using the expression for $h(E)$, we can simplify the expression above

$$R_p(X,s,s') = \frac{x_s}{\beta}\frac{x_{s'}E'(X,s)^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu}E'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}}}{x_s E^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu}E^{\frac{1+\nu}{1+\nu-\alpha}}}, \tag{O3.22}$$

which coincides with the expression for the two-type case.

Step 4: interest rate and risk premium. Using the fact that $R_b(X,s)$ is the risk-neutral expectation of $R_p(X,s,s')$ and $E'(X,s)$ is the risk-neutral expectation of $x_{s'}$, we obtain

$$R_b(X,s) = \left(1-\frac{\alpha}{1+\nu}\right)\frac{x_s}{\beta}\frac{E'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}}}{x_s E^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu}E^{\frac{1+\nu}{1+\nu-\alpha}}}, \tag{O3.23}$$

which coincides with the expression for the two-type case.

The expected return on the surplus claim is given by

$$\mathbb{E}_s[R_p(X,s,s')] = \frac{x_s}{\beta}\frac{\mathbb{E}_s[x_{s'}]E'(X,s)^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu}E'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}}}{x_s E^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu}E^{\frac{1+\nu}{1+\nu-\alpha}}}. \tag{O3.24}$$

Taking the difference of the previous two equations, we obtain the risk premium on the surplus claim:

$$\mathbb{E}_s[R_p^e(X,s,s')] = \frac{x_s}{\beta}\frac{[\mathbb{E}_s[x_{s'}] - E'(X,s)]E'(X,s)^{\frac{\alpha}{1+\nu-\alpha}}}{x_s E^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu}E^{\frac{1+\nu}{1+\nu-\alpha}}}. \tag{O3.25}$$

Step 5: law of motion of aggregate state variables. The risk-neutral probability is given by

$$\frac{p_{ss'}\Lambda(X,s,s')}{\sum_{\tilde{s}\in\mathcal{S}} p_{s\tilde{s}}\Lambda(X,s,\tilde{s})} = \frac{p_{ss'}(X)R_p^{-1}(X,s,s')}{\sum_{\tilde{s}\in\mathcal{S}} p_{s\tilde{s}}(X)R_p^{-1}(X,s,\tilde{s})} = \frac{p_{ss'}(X)\left[x_{s'} - \frac{\alpha}{1+\nu}E'(X,s)\right]^{-1}}{\sum_{\tilde{s}\in\mathcal{S}} p_{s\tilde{s}}(X)\left[x_{\tilde{s}} - \frac{\alpha}{1+\nu}E'(X,s)\right]^{-1}}. \tag{O3.26}$$

From the law of motion of $\mathcal{L}$, we obtain

$$E'(X,s) = \sum_{s'\in\mathcal{S}} x_{s'}\frac{p_{ss'}(X)\left[x_{s'} - \frac{\alpha}{1+\nu}E'(X,s)\right]^{-1}}{\sum_{\tilde{s}\in\mathcal{S}} p_{s\tilde{s}}(X)\left[x_{\tilde{s}} - \frac{\alpha}{1+\nu}E'(X,s)\right]^{-1}}. \tag{O3.27}$$

Rearranging the expression above, we obtain

$$\sum_{s'\in\mathcal{S}}\frac{p_{ss'}(X)(x_{s'} - E'(X,s))}{x_{s'} - \frac{\alpha}{1+\nu}E'(X,s)} = 0 \tag{O3.28}$$

The left-hand side is positive for $E'(X,s) = x^1$, it is negative for $E'(X,s) = x^N$, and it is strictly decreasing in $E'(X,s)$, assuming the condition $x^N < \frac{x^1}{\alpha}$ such that the denominator is positive in the range $x^1 < E'(X,s) < x^N$. Therefore, a solution exists and it is unique.

The law of motion of the wealth share is given by

$$\eta_i'(X,s,s') = \frac{\eta_i R_{i,n}(X,s,s')}{\sum_{j=1}^I \eta_j R_{j,n}(X,s,s')} = \eta_i\frac{R_{i,n}(X,s,s')}{R_p(X,s,s')} = \eta_i\frac{p_{ss'}}{p_{ss'}(X)}. \tag{O3.29}$$

□

O3.3 Proof of Proposition O.3

Proof. We will construct an equilibrium that has iid returns for any financial asset. We guess-and-verify that the consumption-wealth ratio and the net-worth multiplier are constant.

Step 1: consumption and portfolio decisions. Let the consumption-wealth ratio be given by $c_i^*(X,s) = 1-\beta^*$, given a constant β^* that we need to determine. Given that there is no heterogeneity in beliefs, we obtain from the market clearing condition for Arrow securities $R_{i,n}^*(X,s,s') = R_p^*(X,s,s')$. Plugging $c_i^*(X,s)$ and $R_{i,n}^*(X,s,s')$ into the expression for $\Lambda_i^*(X,s,s')$, we obtain

$$\Lambda_i^*(X,s,s') = \beta^\theta(\beta^*)^{-\frac{\theta}{\psi}}[R_p^*(X,s,s')]^{-\gamma}. \tag{O3.30}$$

Step 2: net-worth multiplier. From the envelope condition, we obtain

$$v_i^*(X,s)^{1-\psi^{-1}} = (1-\beta)c_i^*(X,s)^{-\psi^{-1}} \Rightarrow v_i^*(X,s) = (1-\beta)^{\frac{1}{1-\psi^{-1}}}(1-\beta^*)^{-\frac{\psi^{-1}}{1-\psi^{-1}}}. \tag{O3.31}$$

Step 3: wages, hours, and profits. Using $\alpha = \hat{\alpha}\epsilon$, $\xi = \hat{\xi}\epsilon$, and taking the limit of the expressions for wages, hours, and profits as $\epsilon \to 0$, we obtain the expressions provided in the proposition.

Step 4: The price and return on the surplus claim. For an arbitrary α and ξ, the market clearing condition for goods implies that

$$P^*(X,s) = \left.\frac{x_s E^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu}E^{\frac{1+\nu}{1+\nu-\alpha}}}{1-\beta^*}\right|_{\epsilon=0} = \frac{x_s}{1-\beta^*}. \tag{O3.32}$$

The return on the surplus claim is given by

$$R_p^*(X,s,s') = \frac{x_s}{\beta^*}\left.\frac{x_{s'}E'(X,s)^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu}E'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}}}{x_s E^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu}E^{\frac{1+\nu}{1+\nu-\alpha}}}\right|_{\epsilon=0} = \frac{x_{s'}}{\beta^*}. \tag{O3.33}$$

Step 4: The economy's SDF. From the pricing equation, we obtain

$$\mathbb{E}_i[\Lambda_i^*(X,s,s')R_p^*(X,s,s')] = 1 \Rightarrow \beta^\theta(\beta^*)^{-\theta}\mathbb{E}_i[(x_s')^{1-\gamma}] = 1 \tag{O3.34}$$

Rearranging the expression above, we obtain

$$\beta^* = \beta\mathbb{E}_i[(x_s')^{1-\gamma}]^{\frac{1-\psi^{-1}}{1-\gamma}}. \tag{O3.35}$$

Notice that the condition $\beta^* < 1$ is required to ensure that the consumption-wealth ratio is positive.

The SDF is then given by

$$\Lambda^*(X,s,s') = \beta\mathbb{E}^*[x_{s'}^{1-\gamma}]^{\frac{\gamma-\psi^{-1}}{1-\gamma}}(x_{s'})^{-\gamma}, \tag{O3.36}$$

using the fact that $\Lambda^*(X,s,s') = \Lambda_i^*(X,s,s')$.

Step 6: Law of motion of aggregate state variables. The risk-neutral probability is given by

$$\frac{p^*_{s'}\Lambda^*(X,s,s')}{\sum_{\tilde{s}\in\mathcal{S}} p^*_{\tilde{s}}\Lambda^*(X,s,\tilde{s})} = \frac{p^*_{s'}x_{s'}^{-\gamma}}{\mathbb{E}^*[x_{s'}^{-\gamma}]} \tag{O3.37}$$

Hence, $E'(X,s)$ is given by

$$E'(X,s) = \frac{\mathbb{E}^*[x_{s'}^{1-\gamma}]}{\mathbb{E}^*[x_{s'}^{-\gamma}]}. \tag{O3.38}$$

Step 5: The interest rate and risk premium on surplus claim. The interest rate and risk premium are given by

$$R_b^*(X,s) = \frac{E'(X,s)}{\beta^*}, \qquad R_p^e(X,s) = \frac{\mathbb{E}^*[x_{s'}] - E'(X,s)}{\beta^*}. \tag{O3.39}$$

Using the expression for β^* and $E'(X,s)$, we can write the interest rate as follows

$$R_b^*(X,s) = \beta^{-1}\mathbb{E}^*[(x_s')^{1-\gamma}]^{\frac{\psi^{-1}-\gamma}{1-\gamma}}\mathbb{E}^*[(x_{s'})^{-\gamma}]^{-1}, \tag{O3.40}$$

The expected return on the surplus claim is given by

$$\mathbb{E}^*\left[R_p^*(X,s,s')\right] = \beta^{-1}\mathbb{E}^*[(x_s')^{1-\gamma}]^{\frac{\psi^{-1}-1}{1-\gamma}}\mathbb{E}^*[x_{s'}] \tag{O3.41}$$

and the risk premium on the surplus claim is given by

$$\mathbb{E}^*\left[\frac{R_p^*(X,s,s')}{R_b^*(X,s)}\right] = \frac{\mathbb{E}^*[x_{s'}]\mathbb{E}^*[x_{s'}^{-\gamma}]}{\mathbb{E}^*[x_{s'}^{1-\gamma}]} \tag{O3.42}$$

□

O3.4 Proof of Proposition O.4

Proof. We provide a characterization of the first-order correction for the value function, summarized by the net-worth multiplier $v_i(X,s;\epsilon)$, and the policy functions, namely the consumption-wealth ratio $c_i(X,s;\epsilon)$ and the portfolio return $R_{i,n}(X,s,s';\epsilon)$, given the expansion for the economy's SDF

$$\Lambda(X,s,s';\epsilon) = \Lambda^*(X,s,s') + \hat{\Lambda}(X,s,s')\epsilon + \mathcal{O}(\epsilon), \tag{O3.43}$$

where $\hat{\Lambda}(X,s,s')$ is the first-order correction for the SDF. We take $\hat{\Lambda}(X,s,s')$ as given for now and we will solve for it in a later stage.

Step 1: value function. The Bellman equation for household i can be written as follows:

$$\frac{v_i(X,s;\epsilon)^{1-\psi^{-1}}}{1-\psi^{-1}} = (1-\beta)\frac{c_i^{1-\psi^{-1}}}{1-\psi^{-1}} + \beta\frac{\left[\sum_{s'\in\mathcal{S}} p^i_{ss'}(v_i(X',s';\epsilon)R'_n(1-c_i))^{1-\gamma}\right]^{\frac{1-\psi^{-1}}{1-\gamma}}}{1-\psi^{-1}} \tag{O3.44}$$
$$+ \mu(X,s;\epsilon)\left[1-\sum_{s'\in\mathcal{S}} p^*_{s'}\Lambda(X,s,s';\epsilon)R'_n\right],$$

where $X' = \chi(X,s,s';\epsilon)$.

Taking the derivative of the expression above with respect to ϵ and evaluating at $\epsilon = 0$, we obtain a condition involving the first-order correction for v_i:

$$(v_i^*)^{-\psi^{-1}}\hat{v}_i(X,s) = \beta\left[\sum_{s'\in\mathcal{S}} p^*_{s'}(v_i^* R^*_p(s')\beta^*)^{1-\gamma}\right]^{\frac{\gamma-\psi^{-1}}{1-\gamma}}\left[\sum_{s'\in\mathcal{S}}\delta^i_{ss'}\frac{(v_i^* R^*_p(s')\beta^*)^{1-\gamma}}{1-\gamma} + \right.$$
$$\left.\sum_{s'\in\mathcal{S}} p^*_{s'}(v_i^*)^{-\gamma}(R^*_p(s')\beta^*)^{1-\gamma}\hat{v}_i(X^*,s')\right] - \mu^*(X,s)\sum_{s'\in\mathcal{S}} p^*_{s'}\hat{\Lambda}_1(X,s,s')R^*_p(s'), \tag{O3.45}$$

where we used the fact that $R'_n(X,s,s';0) = R^*_p(X,s,s')$, $c_i(X,s;0) = 1-\beta^*$. We also used the fact that $\chi(X,s,s';0) = X^*$, where $X^* = (E^*, \{\eta_i\}_{i=1}^I)$ as $E'(x,s) = E^*$ and the wealth distribution is constant in the benchmark economy.

Using the results for the benchmark economy, we can simplify the expression above:

$$\hat{v}_i(X,s) = \beta\left[\sum_{s'\in\mathcal{S}} p^*_{s'}x_{s'}^{1-\gamma}\right]^{\frac{\gamma-\psi^{-1}}{1-\gamma}}\left[v_i^*\sum_{s'\in\mathcal{S}}\delta^i_{ss'}\frac{x_{s'}^{1-\gamma}}{1-\gamma} + \sum_{s'\in\mathcal{S}} p^*_{s'}x_{s'}^{1-\gamma}\hat{v}_i(X^*,s')\right]$$
$$- \mu^*(X,s)(v_i^*)^{\psi^{-1}}\sum_{s'\in\mathcal{S}} p^*_{s'}\hat{\Lambda}_1(X,s,s')R_p(s'), \tag{O3.46}$$

where we used the fact that $v^*(X,s)$ is constant and $R^*_p(s')\beta^* = x_{s'}$.

Let's solve for $\mu^*(X,s)$ next. The first-order condition for $R_n(X,s,s';\epsilon)$ is given by

$$\beta\left[\sum_{s'\in\mathcal{S}} p^i_{ss'}(v_i(X',s';\epsilon)R'_n(1-c_i))^{1-\gamma}\right]^{\frac{\gamma-\psi^{-1}}{1-\gamma}} p_{ss'}(v_i(X',s')(1-c_i))^{1-\gamma}R_n(X,s,s')^{-\gamma} = \mu(X,s;\epsilon)p^*_{s'}\Lambda(X,s,s';\epsilon) \tag{O3.47}$$

Multiplying by $R_n(X,s,s')$ both sides and adding across states, we obtain

$$\mu(X,s;\epsilon) = \beta\left[\sum_{s'\in\mathcal{S}} p^i_{ss'}(v_i(X',s';\epsilon)R'_n(1-c_i))^{1-\gamma}\right]^{\frac{1-\psi^{-1}}{1-\gamma}}. \tag{O3.48}$$

Evaluating the expression above at $\epsilon = 0$, we obtain

$$\mu^*(X,s) = \beta v^*(X,s)^{1-\psi^{-1}} \left[\sum_{s'\in\mathcal{S}} p^*_{s'} x_{s'}^{1-\gamma}\right]^{\frac{1-\psi^{-1}}{1-\gamma}}. \tag{O3.49}$$

Given $\mu^*(X,s)$, we obtain a system of equations for $\hat{v}_i(X,s')$:

$$\frac{\hat{v}_i(X,s)}{v^*_i(X,s)} - \beta\mathbb{E}^*[x_{s'}^{1-\gamma}]^{\frac{1-\psi^{-1}}{1-\gamma}} \sum_{s'\in\mathcal{S}} \omega^*_{s'} \frac{\hat{v}_i(X^*,s')}{v^*(X,s)} = \beta\mathbb{E}^*[x_{s'}^{1-\gamma}]^{\frac{1-\psi^{-1}}{1-\gamma}} \left[\sum_{s'\in\mathcal{S}} \frac{\omega^*_{s'}}{1-\gamma}\frac{\delta^i_{ss'}}{p^*_{s'}} - \sum_{s'\in\mathcal{S}} \omega^*_{s'} \frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')}\right] \tag{O3.50}$$

using the fact that $R_p(s')\Lambda^*(X,s,s') = \frac{x_{s'}^{1-\gamma}}{\mathbb{E}^*[x_{s'}^{1-\gamma}]}$ and the definition $\omega^*_s \equiv \frac{p^*_s x_s^{1-\gamma}}{\mathbb{E}^*[x_{s'}^{1-\gamma}]}$.

We will solve first for the case $X = X^*$. We can write the system above in matrix form:

$$\begin{bmatrix} 1-\chi_v\omega^*_1 & -\chi_v\omega^*_2 & \dots & -\chi_v\omega^*_N \\ -\chi_v\omega^*_1 & 1-\chi_v\omega^*_2 & \dots & -\chi_v\omega^*_N \\ \vdots & \vdots & \dots & \vdots \\ -\chi_v\omega^*_1 & -\chi_v\omega^*_2 & \dots & 1-\chi_v\omega^*_N \end{bmatrix} \begin{bmatrix} \frac{\hat{v}_i(X^*,1)}{v^*(X,s)} \\ \frac{\hat{v}_i(X^*,2)}{v^*(X,s)} \\ \vdots \\ \frac{\hat{v}_i(X^*,N)}{v^*(X,s)} \end{bmatrix} = \begin{bmatrix} b^v_{i,1}(X^*) \\ b^v_{i,2}(X^*) \\ \vdots \\ b^v_{i,N}(X^*) \end{bmatrix}, \tag{O3.51}$$

where $\chi_v \equiv \beta\mathbb{E}^*[x_{s'}^{1-\gamma}]^{\frac{1-\psi^{-1}}{1-\gamma}}$ and

$$b^v_{i,s}(X^*) \equiv \chi_v \left[\sum_{s'\in\mathcal{S}} \frac{\omega^*_{s'}}{1-\gamma}\frac{\delta^i_{ss'}}{p^*_{s'}} - \sum_{s'\in\mathcal{S}} \omega^*_{s'} \frac{\hat{\Lambda}(X^*,s,s')}{\Lambda^*(X,s,s')}\right]. \tag{O3.52}$$

Let $\omega^* = [\omega^*_1, \omega^*_2, \dots, \omega^*_N]$ denote a row vector, $\hat{v}_i(X) = [\hat{v}_i(X,1), \dots, \hat{v}_i(X,N)]'$ denote a column vector, $b^v_i(X) = \left[b^v_{i,1}(X), \dots, b^v_{i,N}(X)\right]'$ denote a column-vector, and $\mathbf{1}_N$ denote a N-dimensional column vector filled with ones. We can then write the expression above as follows:

$$[I - \chi_v \mathbb{1}_N \omega^*] \frac{\hat{v}_i(X^*)}{v^*_i} = b^v_i(X^*). \tag{O3.53}$$

The matrix on the left-hand side corresponds to the sum of an invertible matrix and rank-one matrix. An application of the Sherman-Morrison formula gives the inverse of this matrix, which gives the solution

$$\hat{v}_i(X^*) = v^*_i(X,s) \left[I + \frac{\chi_v}{1-\chi_v}\mathbb{1}_N\omega^*\right] b^v_i(X^*) \tag{O3.54}$$

The net-worth multiplier at state (X, s) is then given by

$$\frac{\hat{v}_i(X^*, s)}{v_i^*(X, s)} = b_{i,s}^v + \frac{\chi_v}{1 - \chi_v} \sum_{\tilde{s} \in \mathcal{S}} \omega_{\tilde{s}}^* b_{i,\tilde{s}}^v(X^*). \tag{O3.55}$$

Using the expression for $b_{i,s}^v$, we can write the expression above as follows

$$\frac{\hat{v}_i(X^*, s)}{v_i^*(X, s)} = \chi_v \sum_{\tilde{s} \in \mathcal{S}} \left(\mathbf{1}_{\tilde{s}=s} + \frac{\chi_v}{1 - \chi_v} \omega_{\tilde{s}}^* \right) \left[\sum_{s' \in \mathcal{S}} \frac{\omega_{s'}^*}{1 - \gamma} \frac{\delta_{\tilde{s}s'}^i}{p_{s'}^*} - \sum_{s' \in \mathcal{S}} \omega_{s'}^* \frac{\hat{\Lambda}(X^*, \tilde{s}, s')}{\Lambda^*(X, s, s')} \right]. \tag{O3.56}$$

Taking the average of the expression above using the weights ω_s^*, we obtain

$$\sum_{s \in \mathcal{S}} \omega_s^* \frac{\hat{v}_i(X^*, s)}{v_i^*(X, s)} = \frac{\chi_v}{1 - \chi_v} \sum_{\tilde{s} \in \mathcal{S}} \omega_{\tilde{s}}^* \left[\sum_{s' \in \mathcal{S}} \frac{\omega_{s'}^*}{1 - \gamma} \frac{\delta_{\tilde{s}s'}^i}{p_{s'}^*} - \sum_{s' \in \mathcal{S}} \omega_{s'}^* \frac{\hat{\Lambda}(X^*, \tilde{s}, s')}{\Lambda^*(X, s, s')} \right]. \tag{O3.57}$$

The net-worth multiplier at (X, s) is then given by

$$\frac{\hat{v}_i(X, s)}{v_i^*(X, s)} = \chi_v \left[\sum_{s' \in \mathcal{S}} \frac{\omega_{s'}^*}{1 - \gamma} \frac{\delta_{ss'}^i}{p_{s'}^*} - \sum_{s' \in \mathcal{S}} \omega_{s'}^* \frac{\hat{\Lambda}(X, s, s')}{\Lambda^*(X, s, s')} + \sum_{s' \in \mathcal{S}} \omega_{s'}^* \frac{\hat{v}_i(X^*, s')}{v^*(X, s)} \right] \tag{O3.58}$$

We can then write the expression above as follows:

$$\frac{\hat{v}_i(X, s)}{v_i^*(X, s)} = \chi_v \sum_{s' \in \mathcal{S}} \omega_{s'}^* \left[\frac{1}{1 - \gamma} \frac{\delta_{ss'}^i}{p_{s'}^*} - \frac{\hat{\Lambda}(X, s, s')}{\Lambda^*(X, s, s')} \right] + \chi_v \overline{v}, \tag{O3.59}$$

where

$$\overline{v} \equiv \frac{\chi_v}{1 - \chi_v} \sum_{\tilde{s} \in \mathcal{S}} \omega_{\tilde{s}}^* \sum_{s' \in \mathcal{S}} \omega_{s'}^* \left[\frac{1}{1 - \gamma} \frac{\delta_{\tilde{s}s'}^i}{p_{s'}^*} - \frac{\hat{\Lambda}(X^*, \tilde{s}, s')}{\Lambda^*(X^*, \tilde{s}, s')} \right]. \tag{O3.60}$$

Step 2: consumption-wealth ratio. From the envelope condition, the consumption-wealth ratio is given by

$$c_i(X, s; \epsilon) = (1 - \beta)^{\psi} v_i(X, s; \epsilon)^{1 - \psi}. \tag{O3.61}$$

The first-order correction for consumption is then given by

$$\hat{c}_1(X, s) = (1 - \beta)^{\psi} (v_i^*(X, s))^{1 - \psi} (1 - \psi) \frac{\hat{v}_i(X, s)}{v_i^*(X, s)}. \tag{O3.62}$$

Step 3: portfolio return. Using the expression for the Lagrange multiplier, we can write the first-order condition for the portfolio return as follows

$$\frac{p^i_{ss'}}{p^*_{s'}} v_i(X', s')^{1-\gamma} R_n(X, s, s'; \epsilon)^{-\gamma} = \Lambda(X, s, s'; \epsilon) \sum_{s' \in \mathcal{S}} p^i_{ss'} (v_i(X', s'; \epsilon) R_n(X, s, s'; \epsilon))^{1-\gamma} \tag{O3.63}$$

Expanding the expression above in ϵ, we obtain

$$\frac{\delta^i_{ss'}}{p^*_{s'}} + (1-\gamma)\frac{\hat{v}_i(X^*, s')}{v^*(X, s)} - \gamma \frac{\hat{R}_{n,i}(X, s, s')}{R^*_p(X, s, s')} = \frac{\hat{\Lambda}(X, s, s')}{\Lambda^*(X, s, s')} \tag{O3.64}$$

$$+ \sum_{s' \in \mathcal{S}'} \frac{p^*_{s'}(v^*(X, s) R^*_p(s'))^{1-\gamma}}{\sum_{\tilde{s} \in \mathcal{S}} p^*_{\tilde{s}} (v^*(X, s) R^*_p(\tilde{s}))^{1-\gamma}} \left[\frac{\delta^i_{ss'}}{p^*_{s'}} + (1-\gamma) \left(\frac{\hat{v}_i(X^*, s')}{v^*(X, s)} + \frac{\hat{R}_{n,i}(X, s, s')}{R^*_p(X, s, s')} \right) \right] \tag{O3.65}$$

Rearranging the expression above, we obtain

$$\gamma \frac{\hat{R}_{n,i}(X, s, s')}{R^*_p(X, s, s')} + (1-\gamma) \sum_{\tilde{s} \in \mathcal{S}'} \omega^*_{\tilde{s}} \frac{\hat{R}_{n,i}(X, s, \tilde{s})}{R^*_p(X, s, \tilde{s})} = b^R_i(X, s, s'), \tag{O3.66}$$

where

$$b^R_i(X, s, s') \equiv \frac{\delta^i_{ss'}}{p^*_{s'}} + (1-\gamma)\frac{\hat{v}_i(X^*, s')}{v^*(X, s)} - \sum_{\tilde{s} \in \mathcal{S}} \omega^*_{\tilde{s}} \left[\frac{\delta^i_{s\tilde{s}}}{p^*_{\tilde{s}}} + (1-\gamma)\frac{\hat{v}_i(X^*, \tilde{s})}{v^*(X, s)} \right] - \frac{\hat{\Lambda}_1(X, s, s')}{\Lambda^*(X, s, s')} \tag{O3.67}$$

We can write the system above in matrix form:

$$\begin{bmatrix} \gamma + (1-\gamma)\omega^*_1 & (1-\gamma)\omega^*_2 & \dots & (1-\gamma)\omega^* N \\ (1-\gamma)\omega^*_1 & \gamma + (1-\gamma)\omega^*_2 & \dots & (1-\gamma)\omega^*_N \\ \vdots & \vdots & \dots & \vdots \\ (1-\gamma)\omega^*_1 & (1-\gamma)\omega^*_2 & \dots & \gamma + (1-\omega)\omega^*_N \end{bmatrix} \begin{bmatrix} \frac{\hat{R}_{n,i}(X,s,1)}{R^*_p(X,s,1)} \\ \frac{\hat{R}_{n,i}(X,s,2)}{R^*_p(X,s,2)} \\ \vdots \\ \frac{\hat{R}_{n,i}(X,s,N)}{R^*_p(X,s,N)} \end{bmatrix} = \begin{bmatrix} b^R_i(X, s, 1) \\ b^R_i(X, s, 2) \\ \vdots \\ b^R_i(X, s, N) \end{bmatrix} \tag{O3.68}$$

Denote the matrix above by A^* and define the row vector $\omega^* \equiv [\omega^*(s_1), \omega^*(s_2), \dots, \omega^*(s_N)]$ and the column-vector $\mathbb{1}$ with 1 in every entry. We can then write A^* as follows:

$$A^* = \gamma I + (1-\gamma)\mathbb{1}\omega^*. \tag{O3.69}$$

The inverse of A^* is given by

$$(A^*)^{-1} = \frac{1}{\gamma} I - \frac{1-\gamma}{\gamma} \mathbb{1}\omega^*. \tag{O3.70}$$

The portfolio return is then given by

$$\frac{\hat{R}_{n,i}(X,s,s')}{R_p^*(X,s,s')} = \frac{1}{\gamma} b_i^R(X,s,s') + \frac{1-\gamma}{\gamma} \sum_{\tilde{s}\in\mathcal{S}} \omega_{\tilde{s}}^* \frac{\hat{\Lambda}_1(X,s,\tilde{s})}{\Lambda^*(X,s,\tilde{s})}. \tag{O3.71}$$

We can write the expression above as follows:

$$\frac{\hat{R}_{n,i}(X,s,s')}{R_p^*(X,s,s')} = \frac{1}{\gamma} \sum_{\tilde{s}\in\mathcal{S}} \omega_{\tilde{s}}^* \left[\left(\frac{\delta_{ss'}^i}{p_{s'}^*} - \frac{\delta_{s\tilde{s}}^i}{p_{\tilde{s}}^*} \right) - \frac{\hat{\Lambda}_1(X,s,s')}{\Lambda^*(X,s,s')} \right] + \frac{1-\gamma}{\gamma} \sum_{\tilde{s}\in\mathcal{S}} \omega_{\tilde{s}}^* \left[\left(\frac{\hat{v}_i(X^*,s')}{v^*(X,s)} - \frac{\hat{v}_i(X^*,\tilde{s})}{v^*(X,s)} \right) + \frac{\hat{\Lambda}_1(X,s,\tilde{s})}{\Lambda^*(X,s,\tilde{s})} \right]. \tag{O3.72}$$

Notice that we can write the term involving $\hat{v}_i(X,s)$ as follows

$$\frac{\hat{v}_i(X^*,s')}{v^*(X^*,s')} - \sum_{\tilde{s}\in\mathcal{S}} \omega_{\tilde{s}}^* \frac{\hat{v}_i(X^*,\tilde{s})}{v^*(X^*,\tilde{s})} = \chi_v \sum_{\tilde{s}\in\mathcal{S}} \omega_{\tilde{s}}^* \sum_{\tilde{s}'\in\mathcal{S}} \omega_{\tilde{s}'}^* \left[\frac{1}{1-\gamma} \left(\frac{\delta_{s'\tilde{s}'}^i}{p_{\tilde{s}'}^*} - \frac{\delta_{\tilde{s}\tilde{s}'}^i}{p_{\tilde{s}'}^*} \right) - \left(\frac{\hat{\Lambda}(X^*,s',\tilde{s}')}{\Lambda^*(X^*,s',\tilde{s}')} - \frac{\hat{\Lambda}(X^*,\tilde{s},\tilde{s}')}{\Lambda^*(X^*,\tilde{s},\tilde{s}')} \right) \right] \tag{O3.73}$$

□

O3.5 Proof of Proposition O.5

Proof. From the expression for wages, we obtain:

$$w(E;\epsilon) = \xi \left(\frac{\hat{\alpha}E}{\hat{\xi}} \right)^{\frac{\nu}{1+\nu-\alpha}} = \hat{\xi} \left(\frac{\hat{\alpha}E}{\hat{\xi}} \right)^{\frac{\nu}{1+\nu}} \epsilon + \mathcal{O}(\epsilon^2). \tag{O3.74}$$

Hours are given by

$$h(E;\epsilon) = \exp\left[\frac{1}{1+\nu-\alpha} \log\left(\frac{\hat{\alpha}E}{\hat{\xi}} \right) \right] = \left(\frac{\hat{\alpha}E}{\hat{\xi}} \right)^{\frac{1}{1+\nu}} + \left(\frac{\hat{\alpha}E}{\hat{\xi}} \right)^{\frac{1}{1+\nu}} \frac{\log\left(\frac{\hat{\alpha}E}{\hat{\xi}} \right)}{(1+\nu)^2} \hat{\alpha}\epsilon + \mathcal{O}(\epsilon^2). \tag{O3.75}$$

Profits are given by

$$\pi(X,s;\epsilon) = \left(\frac{\hat{\alpha}}{\hat{\xi}} \right)^{\frac{\alpha}{1+\nu-\alpha}} \left[x_s E^{\frac{\alpha}{1+\nu-\alpha}} - \alpha E^{\frac{1+\nu}{1+\nu-\alpha}} \right] = x_s + \left[x_s \frac{\log(\hat{\alpha}E/\hat{\xi})}{1+\nu} - E \right] \hat{\alpha}\epsilon + \mathcal{O}(\epsilon^2), \tag{O3.76}$$

where we used the following Taylor expansion:

$$E^{\frac{\alpha}{1+\nu-\alpha}} = 1 + \frac{\log E}{1+\nu} \hat{\alpha}\epsilon + \mathcal{O}(\epsilon^2). \tag{O3.77}$$

□

O3.6 Proof of Proposition O.6

Proof. We derive next the expression for the price and return for the surplus claim and riskless asset.

Step 1: price of surplus claim. The market clearing for consumption can be written as

$$P(X,s;\epsilon)\sum_{i=1}^{I}\eta_i c_i(X,s;\epsilon)=\left(\frac{\alpha}{\xi}\right)^{\frac{\alpha}{1+\nu-\alpha}}\left[x_s E^{\frac{\alpha}{1+\nu-\alpha}}-\frac{\alpha}{1+\nu}E^{\frac{1+\nu}{1+\nu-\alpha}}\right]. \tag{O3.78}$$

Expanding the expression above in ϵ, we obtain

$$\frac{\hat{P}(X,s)}{P^*(X,s)}+\sum_{i=1}^{I}\eta_i\frac{\hat{c}_i(X,s)}{c^*(X,s)}=\left[\frac{\log(\hat{\alpha}E/\hat{\xi})}{1+\nu}-\frac{1}{1+\nu}\frac{E}{x_s}\right]\hat{\alpha}. \tag{O3.79}$$

Rearranging the expression above, and using the expression for $\hat{c}_i(X,s)$, we obtain

$$\frac{\hat{P}(X,s)}{P^*(X,s)}=\left[\log(\hat{\alpha}E/\hat{\xi})-\frac{E}{x_s}\right]\frac{\hat{\alpha}}{1+\nu}-(1-\psi)\sum_{i=1}^{I}\eta_i\frac{\hat{v}_i(X,s)}{v^*(X,s)}. \tag{O3.80}$$

Step 2: return on surplus claim. The return on the surplus claim is defined as follows

$$R_p(X,s,s';\epsilon)=\frac{x_s P(\chi(X,s,s';\epsilon),s';\epsilon)}{P(X,s;\epsilon)-\left(x_s h(E)^{\alpha}-\xi\frac{h(E;\epsilon)^{1+\nu}}{1+\nu}\right)}. \tag{O3.81}$$

Expanding the expression above in ϵ, we obtain

$$\frac{\hat{R}_p(X,s,s')}{R_p^*(X,s,s')}=\frac{\hat{P}(X^*,s')}{P^*(X^*,s')}-\left[\frac{P^*(X,s)}{P^*(X,s)-x_s}\frac{\hat{P}(X,s)}{P^*(X,s)}-\frac{1}{P^*(X,s)-x_s}\left(x_s\log\left(\frac{\hat{\alpha}}{\hat{\xi}}E\right)-E\right)\frac{\hat{\alpha}}{1+\nu}\right]. \tag{O3.82}$$

We can write the expression above as follows:

$$\frac{\hat{R}_p(X,s,s')}{R_p^*(X,s,s')}=\frac{\hat{P}(X^*,s')}{P^*(X^*,s')}-\left[(\beta^*)^{-1}\frac{\hat{P}(X,s)}{P^*(X,s)}+(1-(\beta^*)^{-1})\left(\log\left(\frac{\hat{\alpha}}{\hat{\xi}}E\right)-\frac{E}{x_s}\right)\frac{\hat{\alpha}}{1+\nu}\right], \tag{O3.83}$$

using the fact that $P^*(X,s)=x_s/(1-\beta^*)$.

Using the expression for the price of the surplus claim, we obtain

$$\frac{\hat{R}_p(X,s,s')}{R_p^*(X,s,s')}=\left[\log(E^*/E)-\left(\frac{E^*}{x_{s'}}-\frac{E}{x_s}\right)\right]\frac{\hat{\alpha}}{1+\nu}-(1-\psi)\sum_{i=1}^{I}\eta_i\left[\frac{\hat{v}_i(X^*,s')}{v^*(X^*,s')}-\frac{1}{\beta^*}\frac{\hat{v}_i(X,s)}{v^*(X,s)}\right], \tag{O3.84}$$

Step 3: interest rate. The interest rate is given by

$$R_b(X,s,s';\epsilon) = \left[\sum_{s'\in\mathcal{S}} p^*_{s'}\Lambda(X,s,s';\epsilon)\right]^{-1} \Rightarrow \frac{\hat{R}_b(X,s)}{R^*_b(X,s)} = -\sum_{s'\in\mathcal{S}} \frac{p_{s'}x_{s'}^{-\gamma}}{\mathbb{E}^*[x_{s'}^{-\gamma}]}\frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')}. \tag{O3.85}$$

Step 4: stock prices. Stock prices, normalized by current productivity, satisfy the functional equation:

$$Q(X,s;\epsilon) = \sum_{s'\in\mathcal{S}} p^*_{s'}\Lambda(X,s,s';\epsilon)\left[\left(\frac{\hat{\alpha}}{\hat{\xi}}\right)^{\frac{\alpha}{1+\nu-\alpha}}\left(x_{s'}E'(X,s;\epsilon)^{\frac{\alpha}{1+\nu-\alpha}} - \alpha E'(X,s;\epsilon)^{\frac{1+\nu}{1+\nu-\alpha}}\right) + x_{s'}Q(\chi(X,s,s';\epsilon),s';\epsilon)\right] \tag{O3.86}$$

For $\epsilon = 0$, we obtain

$$Q^*(X,s) = \sum_{s'\in\mathcal{S}} p^*_{s'}\Lambda^*(X,s,s')x_{s'}\left[1 + Q^*(X^*,s')\right]. \tag{O3.87}$$

We can write the expression above as follows:

$$Q^*(X,s) = \beta^*\sum_{s'\in\mathcal{S}} \frac{p^*_{s'}x_{s'}^{1-\gamma}}{\mathbb{E}^*[x_{s'}^{1-\gamma}]}\left[1 + Q^*(X^*,s')\right] \Rightarrow Q^*(X,s) = \frac{\beta^*}{1-\beta^*}. \tag{O3.88}$$

Expanding the expression for $Q(X,s)$, we obtain

$$\frac{\hat{Q}(X,s)}{Q^*(X,s)} = \sum_{s'\in\mathcal{S}} \omega^*_{s'}\left[\frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')} + (1-\beta^*)\left(\frac{\log\left(\frac{\hat{\alpha}E^*}{\hat{\xi}}\right)}{1+\nu} - \frac{E^*}{x_{s'}}\right)\hat{\alpha} + \beta^*\frac{\hat{Q}(X^*,s')}{Q(X^*,s)}\right]. \tag{O3.89}$$

Evaluating the expression above at $X = X^*$, we obtain

$$\left[I - \beta^*\mathbb{1}_N\omega^*\right]\hat{Q}(X^*) = b^Q(X^*), \tag{O3.90}$$

where $\hat{Q}(X) \equiv [\hat{Q}(X,1),\ldots,\hat{Q}(X,N)]'$, $b^Q(X) \equiv \left[b^Q(X,1),\ldots,b^Q(X,N)\right]'$, and $b^Q(X,s) \equiv Q^*(X,s)\sum_{s'\in\mathcal{S}}\omega^*_{s'}\left[\frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')} + (1-\beta^*)\left(\frac{\log\left(\frac{\hat{\alpha}E^*}{\hat{\xi}}\right)}{1+\nu} - \frac{E^*}{x_{s'}}\right)\hat{\alpha}\right]$.

Solving the system above, we obtain

$$\hat{Q}(X^*) = \left[I + \frac{\beta^*}{1-\beta^*}\mathbb{1}_N\omega^*\right]b^Q(X^*). \tag{O3.91}$$

We can then write $\hat{Q}(X,s)$ as follows:

$$\frac{\hat{Q}(X,s)}{Q^*(X,s)} = \sum_{s'\in\mathcal{S}} \omega^*_{s'}\left[\frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')} + \beta^*\frac{\hat{\Lambda}(X^*,s,s')}{\Lambda^*(X^*,s,s')} + \frac{(\beta^*)^2}{1-\beta^*}\sum_{\tilde{s}\in\mathcal{S}}\omega^*_{\tilde{s}}\frac{\hat{\Lambda}(X^*,\tilde{s},s')}{\Lambda^*(X^*,s,s')}\right] + \left(\frac{\log\left(\frac{\hat{\alpha}E^*}{\hat{\xi}}\right)}{1+\nu} - \sum_{s'\in\mathcal{S}}\omega^*_{s'}\frac{E^*}{x_{s'}}\right)\hat{\alpha}. \tag{O3.92}$$

Step 5: equity returns. Equity returns are given by

$$R_E(X,s,s';\epsilon) = \frac{x_{s'}Q(\chi(X,s,s';\epsilon),s') + \pi(E'(X,s;\epsilon;\epsilon),s')}{Q(X,s;\epsilon)}. \tag{O3.93}$$

Evaluating the expression above at $\epsilon = 0$, we obtain

$$R_E^*(X,s,s') = \frac{x_{s'}Q^*(X^*,s') + \pi^*(E^*,s')}{Q^*(X,s;\epsilon)} = \frac{x_{s'}}{\beta^*}. \tag{O3.94}$$

The first-order correction is given by

$$\frac{\hat{R}_E(X,s,s')}{R_E^*(X,s,s')} = \beta^* \frac{\hat{Q}(X^*,s')}{Q^*(X,s)} + (1-\beta^*)\frac{\hat{\pi}(E^*,s')}{x_{s'}} - \frac{\hat{Q}(X,s)}{Q^*(X,s)}. \tag{O3.95}$$

Step 6: conditional risk premium. The conditional risk premium is defined as follows:

$$\overline{R}_E(X,s;\epsilon) = \sum_{s'\in\mathcal{S}} p_{s'}^* \left[\frac{R_E(X,s,s';\epsilon)}{R_b(X,s;\epsilon)}\right]. \tag{O3.96}$$

The first-order correction is given by

$$\frac{\widehat{\overline{R}}_E(X,s)}{\overline{R}^*(X,s)} = \sum_{s'\in\mathcal{S}} \frac{p_{s'}^* x_{s'}}{\mathbb{E}^*[x_{s'}]} \frac{\hat{R}_E(X,s,s')}{R_E^*(X,s,s')} - \frac{\hat{R}_b(X,s)}{R_b^*(X,s)}. \tag{O3.97}$$

□

O3.7 Proof of Proposition O.7

Proof. We consider next the law of motion of η_i and $\mathcal{L}$.

Step 1: wealth distribution. The law of motion of η_i can be written as

$$\eta_i'(X,s,s';\epsilon)\sum_{j=1}^{I}\eta_j R_{j,n}(X,s,s';\epsilon)(1-c_j(X,s;\epsilon)) = \eta_i R_{i,n}(X,s,s';\epsilon)(1-c_i(X,s;\epsilon)). \tag{O3.98}$$

Expanding the expression above in ϵ, we obtain

$$\hat{\eta}_i'(X,s,s') = \eta_i\left[\frac{\hat{R}_{i,n}(X,s,s')}{R_{i,n}^*(X,s,s')} - \sum_{j=1}^{I}\eta_i\frac{\hat{R}_{j,n}(X,s,s')}{R_{j,n}^*(X,s,s')} - \frac{c_i^*(X,s)}{1-c_i^*(X,s)}\left(\frac{\hat{c}_i(X,s)}{c_i^*(X,s)} - \sum_{j=1}^{I}\eta_j\frac{\hat{c}_j(X,s)}{c_j^*(X,s)}\right)\right]. \tag{O3.99}$$

Using $c_i^*(X,s) = 1-\beta^*$ gives the expression in the proposition.

Step 2: risk-neutral probability of productivity growth. The law of motion of $\mathcal{L}$ can be written as

$$E'(X,s;\epsilon) = R_b(X,s;\epsilon) \sum_{s'\in\mathcal{S}} p^*_{s'}\Lambda(X,s,s';\epsilon)x_{s'}. \tag{O3.100}$$

Expanding the expression above in ϵ, we obtain

$$\frac{\hat{E}'(X,s)}{E^*(X,s)} = \frac{\hat{R}_b(X,s)}{R^*_b(X,s)} + \sum_{s'\in\mathcal{S}} \frac{p^*_{s'}\Lambda^*(X,s,s')x_{s'}}{\mathbb{E}^*[\Lambda^*(X,s,s')x_{s'}]}\frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')} \tag{O3.101}$$

We can write the expression above as follows:

$$\frac{\hat{E}'(X,s)}{E^*} = \frac{\hat{R}_b(X,s)}{R^*_b(X,s)} + \sum_{s'\in\mathcal{S}} \frac{p^*_{s'}x^{1-\gamma}_{s'}}{\mathbb{E}^*[x^{1-\gamma}_{s'}]}\frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')}. \tag{O3.102}$$

Using the definition of E^* and ω^*_s, we obtain the expression given in the proposition. □

O3.8 Proof of Proposition O.8

Proof. We consider the derivation of the economy's SDF $\hat{\Lambda}(X,s,s')$.

Step 1: the system of equations. The market clearing for the Arrow security paying off in state s' is given by

$$\sum_{i=1}^{I}\eta_i(1-c_i(X,s;\epsilon))R_{n,i}(X,s,s') = R_p(X,s,s')\sum_{i=1}^{I}\eta_i(1-c_i(X,s;\epsilon)). \tag{O3.103}$$

Expanding the expression above, we obtain

$$\sum_{i=1}^{I}\eta_i\hat{R}_{n,i}(X,s,s') = \hat{R}_p(X,s,s'). \tag{O3.104}$$

Using the expression for $\hat{R}_{n,i}(X,s,s')$ and $\hat{R}_p(X,s,s')$, we obtain

$$\frac{1}{\gamma}\left[\frac{\delta_{ss'}(X)}{p^*_{s'}} - \sum_{\tilde{s}'\in\mathcal{S}}\omega^*_{\tilde{s}'}\frac{\delta_{s\tilde{s}'}(X)}{p^*_{\tilde{s}'}} - \frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')}\right] + \frac{1-\gamma}{\gamma}\left[\sum_{i=1}^{I}\eta_i\left[\frac{\hat{v}_i(X^*,s')}{v^*(X,s)} - \sum_{\tilde{s}\in\mathcal{S}}\omega^*_{\tilde{s}}\frac{\hat{v}_i(X^*,\tilde{s})}{v^*(X,s)}\right] + \sum_{\tilde{s}\in\mathcal{S}}\omega^*_{\tilde{s}}\frac{\hat{\Lambda}(X,s,\tilde{s})}{\Lambda^*(X,s,\tilde{s})}\right] =$$
$$\left[\log\frac{E^*}{E} - \left(\frac{E^*}{x_{s'}} - \frac{E}{x_s}\right)\right]\frac{\hat{\alpha}}{1+\nu} + (\psi-1)\sum_{i=1}^{I}\eta_i\left[\frac{\hat{v}_i(X^*,s')}{v^*(X^*,s')} - \frac{1}{\beta^*}\frac{\hat{v}_i(X,s)}{v^*(X,s)}\right]. \tag{O3.105}$$

Using the expression for $\hat{v}_i(X,s)$, we obtain

$$\frac{1}{\gamma}\frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')} + \frac{1-\gamma}{\gamma}\left[\beta^* \sum_{\tilde{s}\in\mathcal{S}} \omega^*_{\tilde{s}} \sum_{\tilde{s}'\in\mathcal{S}} \omega^*_{\tilde{s}'} \left(\frac{\hat{\Lambda}(X^*,s',\tilde{s}')}{\Lambda^*(X^*,s',\tilde{s}')} - \frac{\hat{\Lambda}(X^*,\tilde{s},\tilde{s}')}{\Lambda^*(X^*,\tilde{s},\tilde{s}')}\right) - \sum_{\tilde{s}'\in\mathcal{S}} \omega^*_{\tilde{s}'} \frac{\hat{\Lambda}(X,s,\tilde{s}')}{\Lambda^*(X,s,\tilde{s}')}\right] \tag{O3.106}$$

$$(\psi-1)\beta^*\left[-\sum_{\tilde{s}'\in\mathcal{S}} \omega^*_{\tilde{s}'}\left(\frac{\hat{\Lambda}(X^*,s',\tilde{s}')}{\Lambda^*(X^*,s',\tilde{s}')} - \frac{1}{\beta^*}\frac{\hat{\Lambda}(X,s,\tilde{s}')}{\Lambda^*(X,s,\tilde{s}')}\right) + \sum_{\tilde{s}\in\mathcal{S}} \omega^*_{\tilde{s}} \sum_{\tilde{s}'\in\mathcal{S}} \omega^*_{\tilde{s}'} \frac{\hat{\Lambda}(X^*,\tilde{s},\tilde{s}')}{\Lambda^*(X^*,\tilde{s},\tilde{s}')}\right] = b^{\Lambda}(X,s,s'), \tag{O3.107}$$

where

$$b^{\Lambda}(X,s,s') \equiv \frac{1}{\gamma}\left[\frac{\delta_{ss'}(X)}{p^*_{s'}} - \sum_{\tilde{s}'\in\mathcal{S}} \omega^*_{\tilde{s}'}\frac{\delta_{s\tilde{s}'}(X)}{p^*_{\tilde{s}'}}\right] + \frac{\beta^*}{\gamma}\left[\sum_{\tilde{s}\in\mathcal{S}} \omega^*_{\tilde{s}} \sum_{\tilde{s}'\in\mathcal{S}} \omega^*_{\tilde{s}'}\left[\frac{\delta_{s'\tilde{s}'}(X)}{p^*_{\tilde{s}'}} - \frac{\delta_{\tilde{s}\tilde{s}'}(X)}{p^*_{\tilde{s}'}}\right]\right] + \\ -\frac{\psi-1}{1-\gamma}\sum_{\tilde{s}\in\mathcal{S}} \omega^*_{\tilde{s}} \sum_{\tilde{s}'\in\mathcal{S}} \omega^*_{\tilde{s}'}\left[\frac{\delta_{s\tilde{s}'}(X)}{p^*_{\tilde{s}'}} - \beta^*\frac{\delta_{s'\tilde{s}'}(X^*)}{p^*_{\tilde{s}'}} + \beta^*\frac{\delta_{\tilde{s}\tilde{s}'}(X)}{p^*_{\tilde{s}'}}\right] - \left[\log\frac{E^*}{E} - \left(\frac{E^*}{x_{s'}} - \frac{E}{x_s}\right)\right]\frac{\hat{\alpha}}{1+\nu}. \tag{O3.108}$$

We can simplify the expression above as follows:

$$\frac{1}{\gamma}\frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')} + (\psi-\gamma^{-1})\left[\omega^*\cdot\hat{\Lambda}(X,s) - \beta^*\omega^*\cdot\hat{\Lambda}(X^*,s') + \beta^*\sum_{\tilde{s}\in\mathcal{S}}\omega_{\tilde{s}}(\omega^*\cdot\hat{\Lambda}(X^*,\tilde{s}))\right] = b^{\Lambda}(X,s,s'), \tag{O3.109}$$

where $\hat{\Lambda}(X,s) = \left[\frac{\hat{\Lambda}(X^*,s,1)}{\Lambda^*(X^*,s,1)}, \frac{\hat{\Lambda}(X^*,s,2)}{\Lambda^*(X^*,s,2)}, \ldots, \frac{\hat{\Lambda}(X^*,s,N)}{\Lambda^*(X^*,s,N)}\right]'$ and $\omega^*\cdot\hat{\Lambda}(X,s) = \sum_{\tilde{s}'}\omega^*_{\tilde{s}'}\frac{\hat{\Lambda}(X,s,\tilde{s}')}{\Lambda^*(X,s,\tilde{s}')}$.

Step 2: solving the system. We can write the system above in matrix form as follows:

$$\left[\gamma^{-1}I + (\psi-\gamma^{-1})\mathbb{1}_N\omega^*\right]\hat{\Lambda}(X,s) = \tilde{b}^{\Lambda}(X,s), \tag{O3.110}$$

where $\tilde{b}^{\Lambda}(X,s) = [\tilde{b}^{\Lambda}(X,s,1), \tilde{b}^{\Lambda}(X,s,2), \ldots, \tilde{b}^{\Lambda}(X,s,N)]'$ and

$$\tilde{b}^{\Lambda}(X,s,s') = b^{\Lambda}(X,s,s') + (\psi-\gamma^{-1})\beta^*\left[\omega^*\cdot\hat{\Lambda}(X^*,s') - \sum_{\tilde{s}\in\mathcal{S}}\omega_{\tilde{s}}(\omega^*\cdot\hat{\Lambda}(X^*,\tilde{s}))\right] \tag{O3.111}$$

Applying the Sherman-Morrison formula, we can invert the system above

$$\hat{\Lambda}(X,s) = \left[\gamma I - (\gamma-\psi^{-1})\mathbb{1}_N\omega^*\right]\tilde{b}^{\Lambda}(X,s). \tag{O3.112}$$

We can then write the expression above as follows:

$$\frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')} = \gamma b^{\Lambda}(X,s,s') + (\gamma\psi - 1)\beta^* \left[\omega^* \cdot \hat{\Lambda}(X^*,s') - \sum_{\tilde{s}\in\mathcal{S}} \omega_{\tilde{s}}(\omega^* \cdot \hat{\Lambda}(X^*,\tilde{s}))\right] - (\gamma - \psi^{-1})\omega^* b^{\Lambda}(X,s). \tag{O3.113}$$

Step 3: solving for the average $\hat{\Lambda}(X,s,s')$. Assuming $X = X^*$, multiplying by $\omega^*_{s'}$, and adding across states, we obtain

$$\omega^* \hat{\Lambda}(X^*,s) = \psi^{-1}\omega^* b^{\Lambda}(X^*,s). \tag{O3.114}$$

Averaging across s, we obtain

$$\sum_{\tilde{s}\in\mathcal{S}} \omega^*_{\tilde{s}}[\omega^* \hat{\Lambda}(X^*,\tilde{s})] = \psi^{-1} \sum_{\tilde{s}\in\mathcal{S}} \omega^*_{\tilde{s}}[\omega^* b^{\Lambda}(X^*,\tilde{s})]. \tag{O3.115}$$

We can then write $\hat{\Lambda}(X,s,s')$ as follows

$$\frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')} = \gamma b^{\Lambda}(X,s,s') - (\gamma - \psi^{-1})\omega^* b^{\Lambda}(X,s) + (\gamma - \psi^{-1})\beta^* \left[\omega^* \cdot b^{\Lambda}(X^*,s') - \sum_{\tilde{s}\in\mathcal{S}} \omega_{\tilde{s}}(\omega^* \cdot b^{\Lambda}(X^*,\tilde{s}))\right]. \tag{O3.116}$$

Step 4: simplifying the expression for $b^{\Lambda}(X,s,s')$. We can write $b^{\Lambda}(X,s,s')$ as follows:

$$\begin{aligned} b^{\Lambda}(X,s,s') &= \frac{1}{\gamma}\left[\frac{\delta_{ss'}(X)}{p^*_{s'}} - \omega^* \cdot \delta_s(X)\right] + \frac{\beta^*}{\gamma}\left[\omega^* \delta_{s'}(X) - \sum_{\tilde{s}} \omega^*_{\tilde{s}} \omega^* \cdot \delta_{\tilde{s}}(X)\right] \\ &+ \frac{\psi - 1}{1-\gamma}\left[\omega^* \cdot \delta_s(X) - \beta^* \omega^* \cdot \delta_{s'}(X) + \beta^* \sum_{\tilde{s}} \omega^*_{\tilde{s}}(\omega^* \cdot \delta_{\tilde{s}}(X))\right] - \left[\log\frac{E^*}{E} - \left(\frac{E^*}{x_{s'}} - \frac{E}{x_s}\right)\right]\frac{\hat{\alpha}}{1+\nu}. \end{aligned} \tag{O3.117}$$

Combining terms, we obtain

$$b^{\Lambda}(X,s,s') = \frac{1}{\gamma}\frac{\delta_{ss'}(X)}{p^*_{s'}} - \frac{\psi - \gamma^{-1}}{\gamma - 1}\left[\omega^* \cdot \delta_s(X) - \beta^* \omega^* \cdot \delta_{s'}(X) + \beta^* \sum_{\tilde{s}} \omega^*_{\tilde{s}}(\omega^* \cdot \delta_{\tilde{s}}(X))\right] - \left[\log\frac{E^*}{E} - \left(\frac{E^*}{x_{s'}} - \frac{E}{x_s}\right)\right]\frac{\hat{\alpha}}{1+\nu}. \tag{O3.118}$$

Notice that we can $\hat{\Lambda}(X,s,s')$ as follows:

$$\frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')} = \frac{\hat{\Lambda}(X^*,s,s')}{\Lambda^*(X,s,s')} + \psi^{-1}\left[\log\frac{E}{E^*} - \frac{E - E^*}{x_s}\right]\frac{\hat{\alpha}}{1+\nu}. \tag{O3.119}$$

□

O3.9 The economy with no labor frictions and iid returns

Suppose that labor can be chosen conditional on the current productivity level. In this case, the problem of the firm can be written as

$$\max_{h_t} x_t h_t^{\alpha} - w_t h_t, \tag{O3.120}$$

where $w_t \equiv \frac{W_t}{A_{t-1}}$. Labor demand takes the familiar form:

$$\alpha x_t h_t^{\alpha-1} = w_t \Rightarrow h_t = \left(\frac{\alpha x_t}{w_t}\right)^{\frac{1}{1-\alpha}}. \tag{O3.121}$$

The labor supply from households is given by

$$h_t = \left(\frac{w_t}{\xi}\right)^{\frac{1}{\nu}}. \tag{O3.122}$$

Combining labor supply and labor demand, we obtain the equilibrium hours and wages:

$$h_t = \left(\frac{\alpha x_t}{\xi}\right)^{\frac{1}{1+\nu-\alpha}}, \qquad w_t = \xi^{\frac{1-\alpha}{1+\nu-\alpha}} (\alpha x_t)^{\frac{\nu}{1+\nu-\alpha}}. \tag{O3.123}$$

Firm's profits are given by

$$\pi_t = A_{t-1}(1-\alpha)\left(\frac{\alpha}{\xi}\right)^{\frac{\alpha}{1+\nu-\alpha}} x_t^{\frac{1+\nu}{1+\nu-\alpha}}. \tag{O3.124}$$

Total surplus is given by

$$\tilde{C}_t = A_{t-1}\left[x_t h_t^{\alpha} - \xi \frac{h_t^{1+\nu}}{1+\nu}\right] = A_{t-1}\left(1 - \frac{\alpha}{1+\nu}\right)\left(\frac{\alpha}{\xi}\right)^{\frac{\alpha}{1+\nu-\alpha}} x_t^{\frac{1+\nu}{1+\nu-\alpha}}. \tag{O3.125}$$

Let $P(X,s)$ denote the price of the surplus claim normalized by lagged productivity. Market clearing condition implies that

$$1-\beta^* = \left(1 - \frac{\alpha}{1+\nu}\right)\left(\frac{\alpha}{\xi}\right)^{\frac{\alpha}{1+\nu-\alpha}} \frac{x_s^{\frac{1+\nu}{1+\nu-\alpha}}}{P(X,s)}, \tag{O3.126}$$

where $1-\beta^*$ is the consumption-wealth ratio, which we assume to be constant.

The return on the surplus claim is given by

$$R_p(X, s, s') = \frac{x_s}{\beta^*} \frac{x_{s'}^{\frac{1+\nu}{1+\nu-\alpha}}}{x_s^{\frac{1+\nu}{1+\nu-\alpha}}} = \frac{x_{s'}}{\beta^*} \left(\frac{x_{s'}}{x_s} \right)^{\frac{\alpha}{1+\nu-\alpha}}. \tag{O3.127}$$